An Introduction to Modelling and Control of Systems Governed by PDE's: Analysis

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2.1 Introduction

The aim of this part is to learn the concepts semigroups and infinitesimal generator.

The reason for this is that we want to know if the partial differential equations (pde's) have a (unique) solution.

Note there is a difference between knowing the existence of a solution and having the form/expression of the solution.

Example (Transport equation)

On the spatial domain $[0, 1]$ consider the pde:

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1],\ t \geq 0, \\
x(1, t) &= 0 \\
x(\zeta, 0) &= x_0(\zeta) \qquad \text{(given)}.
\end{aligned}$$

□

Example (Diffusion equation)

On the spatial domain $[0, 1]$ consider the p.d.e.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t), \qquad \zeta \in [0,1],\ t \geq 0, \\ \frac{\partial x}{\partial \zeta}(0, t) &= 0 \\ \frac{\partial x}{\partial \zeta}(1, t) &= 0 \\ x(\zeta, 0) &= x_0(\zeta) \qquad \text{(given)}. \end{aligned}$$

□

2.2 Semigroups

Throughout this course, we assume that

- **X is a (complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$.**
- **$\mathcal{L}(X)$ denotes the set of linear and bounded operators from X to X.**

Definition

A strongly continuous semigroup is an operator-valued function from $[0,\infty)$ to $\mathcal{L}(X)$ which satisfies

- $T(0) = I$
- $T(t)T(s) = T(t+s), \quad t, s \in [0,\infty)$
- **For all $x_0 \in X$ there holds**

$$\lim_{t \downarrow 0} T(t)x_0 = x_0.$$

□

Notation: $(T(t))_{t\geq 0}$; **Short**: C_0-semigroup.

To motivate this definition, think of

$$x_0 \mapsto T(t)x_0$$

as a solution of a time-invariant, linear differential equation.

- $T(0) = I$ **Trivial.**
- **For fixed** t, $T(t)$ **is linear** **Is linearity of diff. eq.**
- $T(t)T(s) = T(t+s)$ **Is time invariance.**
- $T(t)x_0 \to x_0$ **if** $t \downarrow 0$ **Is strong continuity.**

Example

Let A be an $n \times n$-matrix, then e^{At} is a C_0-semigroup on $\mathbb{C}^n$ or $\mathbb{R}^n$.

For instance, if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

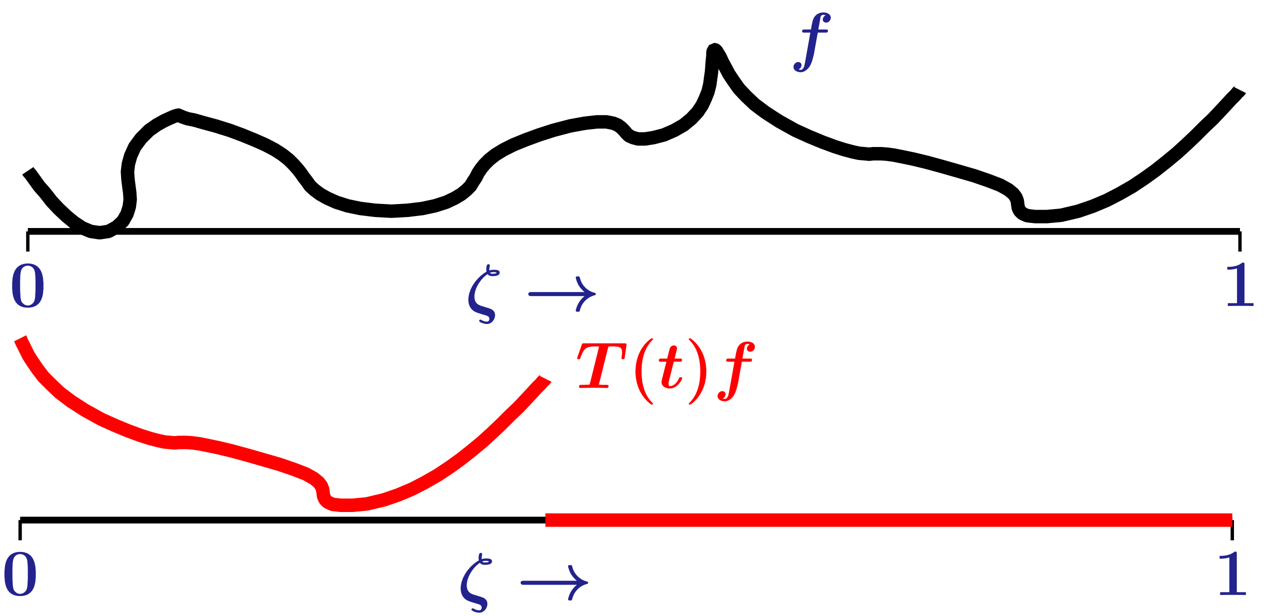
$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

□

Example

Let $X = L^2(0,1)$ and define for $t \geq 0$ and $\zeta \in [0,1]$

$$(T(t)f)(\zeta) = \begin{cases} f(t+\zeta) & \zeta + t \leq 1 \\ 0 & \zeta + t > 1 \end{cases}$$



□

An important property of a C_0-semigroup is that it is exponentially bounded.

Lemma

There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \geq 0$

$$\|T(t)\| \leq Me^{\omega t}.$$

□

The proof is as follows.

- $\sup_{t\in[0,1]} \|T(t)\| \leq M$.

  This follows from the strong continuity and the uniform boundedness theorem.

  Note $M \geq 1$.

- **Choose $t > 1$ and write $t = n + t_0$ with $t_0 \in [0, 1)$.**

$$
\begin{aligned}
\|T(t)\| &= \|T(n+t_0)\| \\
&= \|T(n)T(t_0)\| \\
&\leq \|T(n)\|\,\|T(t_0)\| \\
&\leq M\|T(n)\| \\
&= M\|T(n-1)T(1)\| \\
&\leq M\|T(1)\|\,\|T(n-1)\| \\
&\leq MM\|T(n-1)\| \\
&\leq M^{n+1} = Me^{\log Mn} \\
&\leq Me^{\log Mt}.
\end{aligned}
$$

2.3 Generators

Consider the finite-dimensional semigroup

$$e^{At} = \begin{pmatrix} e^{t} & e^{3t} - e^{t} \\ 0 & e^{3t} \end{pmatrix}.$$

Question

What is A?

Answer

Evaluate the derivative of the semigroup at $t = 0$.

Since $\frac{d}{dt}e^{At} = Ae^{At}$, we have

$$\frac{d}{dt}e^{At} \Big|_{t=0} = A.$$

□

So given the (general) C_0-semigroup $(T(t))_{t\geq 0}$, we could try to find A by differentiating it at $t = 0$. However, in general $(T(t))_{t\geq 0}$ is only (strongly) continuous.

Definition

If the following limit exists,

$$\lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t},$$

then x_0 is in the domain of A, $D(A)$.

Furthermore, for $x_0 \in D(A)$, we define

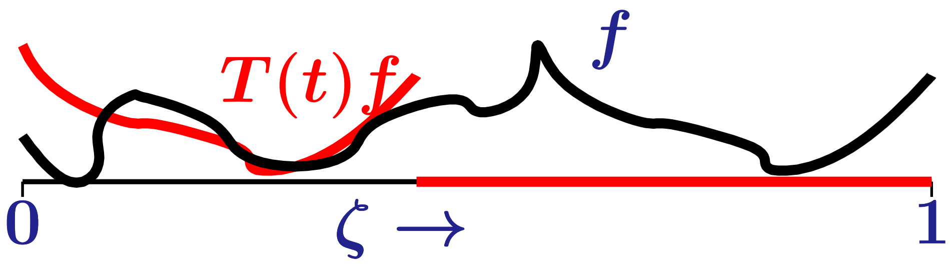
$$Ax_0 = \lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t}.$$

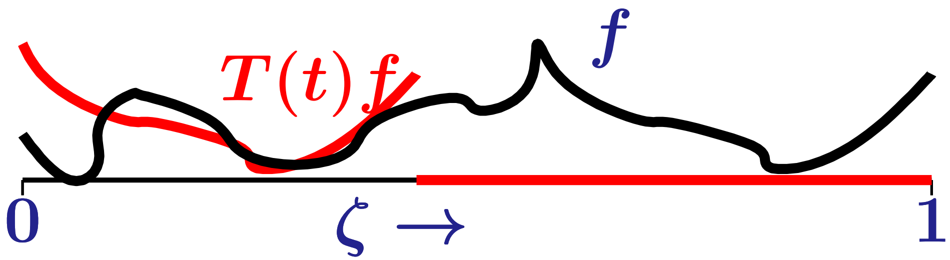
A is named the infinitesimal generator of the C_0-semigroup $(T(t))_{t \geq 0}$. □

Example

Consider the shift semigroup.

$$(T(t)f)(\zeta) = \begin{cases} f(t+\zeta) & \zeta + t \leq 1 \\ 0 & \zeta + t > 1 \end{cases}$$





Choose $\zeta < 1$, then for $t \in (0, \zeta)$, we have that

$$\left(\frac{T(t)f - f}{t}\right)(\zeta) = \frac{f(t+\zeta) - f(\zeta)}{t}.$$

Hence if the limit exists, then f must be differentiable, and the limit is the derivative.

Now take $\zeta = 1$. For any $t > 0$, we have that

$$\left(\frac{T(t)f - f}{t}\right)(1) = \frac{0 - f(1)}{t}.$$

This limit can only exist when $f(1) = 0$.

Concluding, we have that

$$D(A) = \Big\{ f \in L^2(0,1) \mid f \text{ is absolutely continuous, with } \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \Big\}.$$

Furthermore,

$$Af = \frac{df}{d\zeta}.$$

$\square$

Lemma

If $x_0 \in D(A)$, then $T(t)x_0 \in D(A)$, and

$$\frac{d}{dt}[T(t)x_0] = AT(t)x_0.$$

$\square$

Hence for $x_0 \in D(A)$, we have that $x(t) := T(t)x_0$ is a (classical) solution of the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

How does this abstract differential equation relate to a p.d.e.?

We make the following observations:

- **The function x is at every time instant an element of a Hilbert space, i.e., $x(t) \in X$.**
- **Assume that the Hilbert space consists of functions, i.e., $f \in X$ means a.o. $f := \zeta \mapsto f(\zeta)$.**
  For instance, $X = L^2(0, 1)$.

- **This implies that $x(t)$ is for every t a function of ζ. We write $(x(t))(\zeta) = x(\zeta, t)$.**
- **Using this last form we see that**

$$(\dot{x}(t))(\zeta) = \frac{\partial x}{\partial t}(\zeta, t).$$

- **For $A = \frac{d}{d\zeta}$, the abstract differential equation**

$$\dot{x}(t) = Ax(t)$$

becomes

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t).$$

- **In the domain of A are also the boundary conditions:**

$$x(1, t) = 0.$$

- **For a given p.d.e. we (roughly) do the following**
  - **The state x is identified, and the p.d.e. is written as**

$$\frac{\partial x}{\partial t} = \dots$$

  - **The right-hand side defines together with the boundary conditions A.**

2.4 Which A generates a C_0-semigroup?

So we have that every semigroup has an infinitesimal generator, but you would like to know which operator A generates a C_0-semigroup.

The answer is given by the Hille-Yosida Theorem. This theorem we will not treat. We focus on the special case

- $T(t)$ **is a contraction semigroup.**

2.5 Contraction semigroup

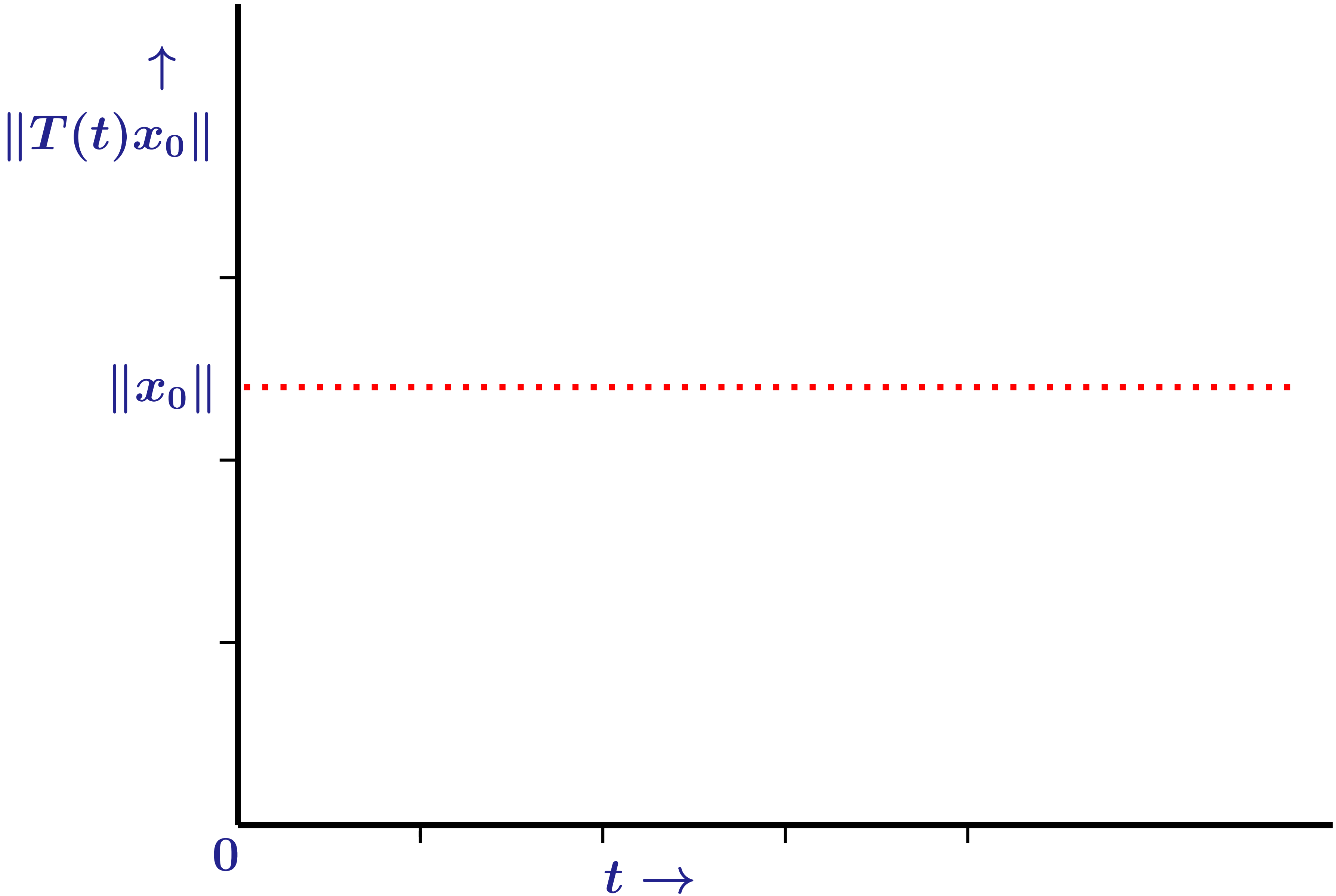
Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is a contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \qquad \text{for all } t \geq 0.$$

□

What can we say about these semigroups?


$\uparrow$
$\|T(t)x_0\|$
$\|x_0\|$
0
$t \rightarrow$

We known that

$$\|T(t)x_0\|^2 = \langle T(t)x_0, T(t)x_0 \rangle.$$

For $x_0 \in D(A)$, we have that the derivative of $T(t)x_0$ equals $AT(t)x_0$.

So if we differentiate $\|T(t)x_0\|^2$, we find

$$\frac{d}{dt}\|T(t)x_0\|^2 = \langle AT(t)x_0, T(t)x_0 \rangle + \langle T(t)x_0, AT(t)x_0 \rangle.$$

At time equal to zero, we find

$$\frac{d}{dt}\left(\|T(t)x_0\|^2\right)\,|_{t=0} = \langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle.$$

We know that at $t = 0$, $\|T(t)x_0\| = \|x_0\|$, and so if $T(t)$ is a contraction semigroup, then

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle = \frac{d}{dt}\|T(t)x_0\|^2\,|_{t=0} \leq 0.$$

This holds for all $x_0 \in D(A)$.

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then A generates a contraction semigroup on X if and only if

1. $\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \leq 0$ **for all** $x_0 \in D(A)$.
2. **The range of** $A - I$ **is the whole of** X.

□

Example

Consider A which is given as

$$Ax = \frac{dx}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

$$D(A) = \Big\{x \in L^2(0,1) \mid x \text{ is absolutely continuous}, \frac{dx}{d\zeta} \in L^2(0,1) \text{ and } x(1) = 0\Big\}.$$

Let us check the properties

- A is densely defined in $L^2(0,1)$.

-

$$
\begin{aligned}
\langle Ax, x\rangle + \langle x, Ax\rangle
&= \int_0^1 \frac{dx}{d\zeta}(\zeta)\overline{x(\zeta)}d\zeta + \int_0^1 x(\zeta)\overline{\frac{dx}{d\zeta}(\zeta)}d\zeta \\
&= \int_0^1 \frac{d}{d\zeta}\left[x(\zeta)\overline{x(\zeta)}\right] d\zeta \\
&= |x(\zeta)|^2\Big|_0^1 \\
&= 0 - |x(0)|^2 \leq 0.
\end{aligned}
$$

- **To see if the range of $(A - I)$ is everything, we have for every $f \in L^2(0,1)$ to solve $(A - I)x = f$.**

  This means solving

$$\frac{dx}{d\zeta}(\zeta) - x(\zeta) = f(\zeta), \qquad \zeta \in (0,1)$$

  with boundary condition $x(1) = 0$.

  The solution of this differential equation with the given boundary value is

$$x(\zeta) = -\int_{\zeta}^{1} e^{\zeta-\tau} f(\tau) d\tau.$$

$\square$

Example

Consider A given by

$$Ax = \frac{d^2x}{d\zeta^2} \qquad \zeta \in [0,1]$$

with the domain

$$D(A) \;=\; \left\{ x \in L^2(0,1) \mid x, \frac{dx}{d\zeta} \text{ are absolutely continuous,} \right. \\ \left. \text{and } \frac{dx}{d\zeta}(0) = 0 = \frac{dx}{d\zeta}(1) \right\}.$$

Clearly A is densely defined, and so it remains to check the other conditions.

We have that

$$
\begin{aligned}
&\langle Ax, x\rangle + \langle x, Ax\rangle \\
&\quad = \int_0^1 \frac{d^2x}{d\zeta^2}(x)\overline{x(\zeta)} + x(\zeta)\overline{\frac{d^2x}{d\zeta^2}}(\zeta)d\zeta \\
&\quad = \left[\frac{dx}{d\zeta}(\zeta)\overline{x(\zeta)} + x(\zeta)\overline{\frac{dx}{d\zeta}}(\zeta)\right]_0^1 - 2\int_0^1 \left|\frac{dx}{d\zeta}(\zeta)\right|^2 d\zeta \\
&\quad \leq 0
\end{aligned}
$$

for $x \in D(A)$**.**

To see that

$$(A - I)x = f$$

is solvable for all $f \in L^2(0,1)$ we have to solve an o.d.e.

The solution is given by

$$x(\zeta) = \cosh(\zeta)x(0) + \int_0^\zeta \sinh(\zeta - \tau)f(\tau)d\tau,$$

with

$$x(0) = \frac{-1}{\sinh(1)} \int_0^1 \cosh(1 - \xi)f(\xi)d\xi.$$

Thus A generates a contraction semigroup.

Now we know that A generates a C_0-semigroup, but can we find this semigroup?

For this we could return to the corresponding p.d.e.

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t), \qquad \zeta \in [0,1],\ t \geq 0,\\
\frac{\partial x}{\partial \zeta}(0,t) &= 0\\
\frac{\partial x}{\partial \zeta}(1,t) &= 0\\
x(\zeta,0) &= x_0(\zeta) \qquad \text{(given).}
\end{aligned}$$

This we can solve by the separation of variables principle. We choose another approach.

- **A is a self-adjoint operator and the inverse of $(A - I)$ is compact.**
- **Hence A has an orthonormal basis of eigenfunctions.**

- **Solving $A\phi_n = \lambda_n \phi_n$ gives**

$$\phi_n(\zeta) = \begin{cases} 1 & \lambda_0 = 0 \\ \sqrt{2}\cos(n\pi\zeta) & \lambda_n = -n^2\pi^2, \qquad n \in \mathbb{N} \end{cases}$$

- **Hence the solution of**

$$\dot{x}(t) = Ax(t), \qquad x(0) = \phi_n$$

is given by

$$x(t) = e^{\lambda_n t}\phi_n$$

- **This must be equal to $T(t)\phi_n$.**

- **Since $\{\phi_n, n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis, we know that**

$$x_0 = \sum_{n=0}^{\infty} \langle x_0, \phi_n \rangle \phi_n$$

Hence

$$\begin{aligned} T(t)x_0 &= T(t)\left(\sum_{n=0}^{\infty} \langle x_0, \phi_n \rangle \phi_n\right) \\ &= \sum_{n=0}^{\infty} \langle x_0, \phi_n \rangle T(t)\phi_n \\ &= \sum_{n=0}^{\infty} \langle x_0, \phi_n \rangle e^{\lambda_n t}\phi_n. \end{aligned}$$

□

2.6 Semigroups and solutions of p.d.e.'s

We have that for any $x_0 \in D(A)$, the function $x(t) := T(t)x_0$ is the solution of

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

How for general x_0?

Note that in general $Ax(t)$ has no meaning and so has $\dot{x}(t)$.

To solve this question we have first to introduce the adjoint of A.

Definition

Let A be a densely defined operator with domain $D(A)$. The domain of A^*, $D(A^*)$, is defined as consisting of those $w \in X$ for which there exists a $v \in X$ such that

$$\langle w, Ax \rangle = \langle v, x \rangle \quad \text{for all } x \in D(A).$$

If $w \in D(A^*)$, then A^* is defined as

$$A^* w = v.$$

A^* is named the adjoint of A. □

Lemma

Let $x_0 \in Z$, and define $x(t) = T(t)x_0$. Then for every $w \in D(A^*)$, there holds that

$$\frac{d}{dt}\langle w, x(t)\rangle = \langle A^*w, x(t)\rangle.$$

□

This implies that $x(t) := T(t)x_0$ is the weak solution of $\dot{x}(t) = Ax(t), x(0) = x_0$.

2.7 Summary

We have seen the following

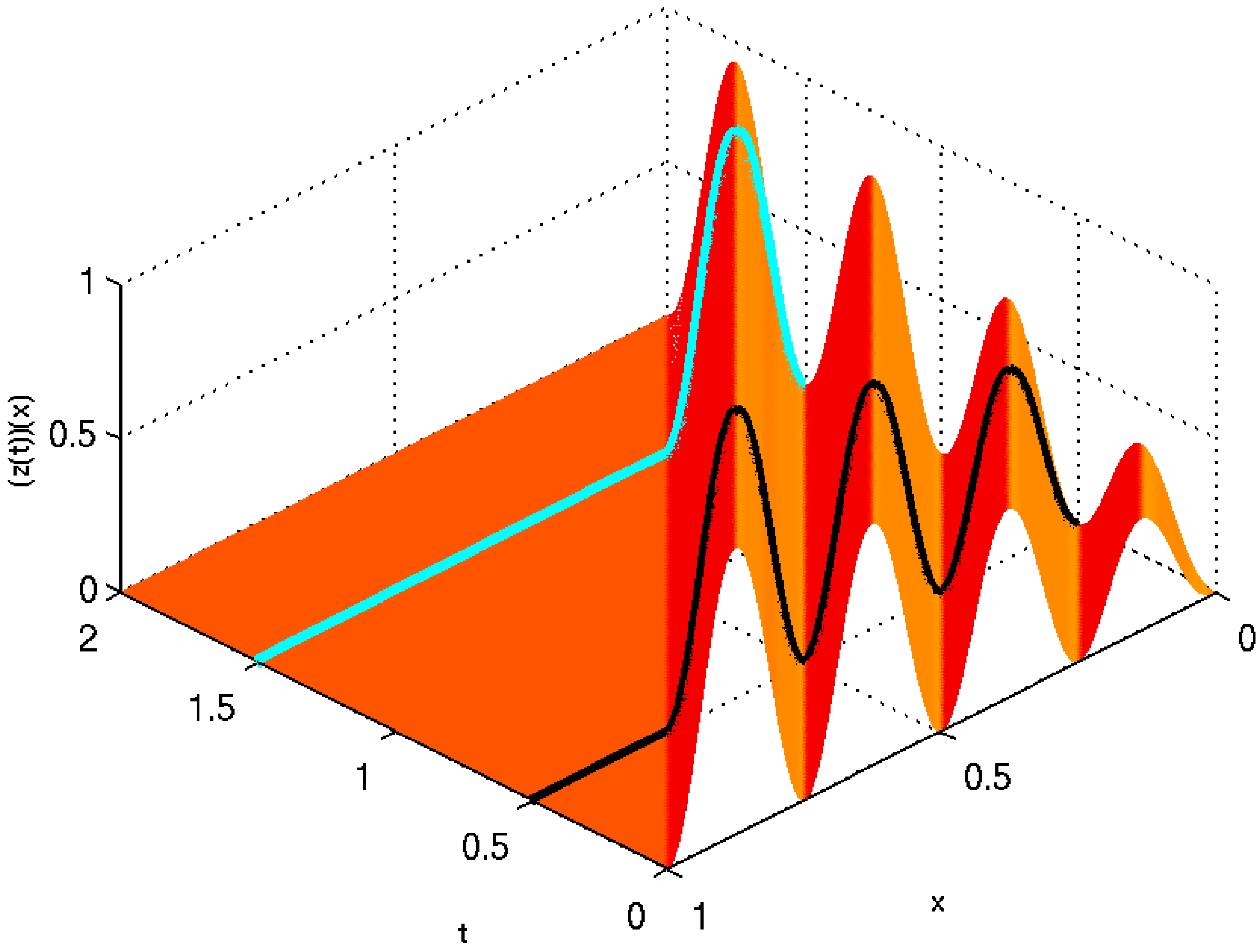
- **A p.d.e. like**

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t) \\ x(1, t) &= 0 \end{aligned}$$

will be written as

$$\dot{x}(t) = Ax(t)$$

where x is the state, see figure

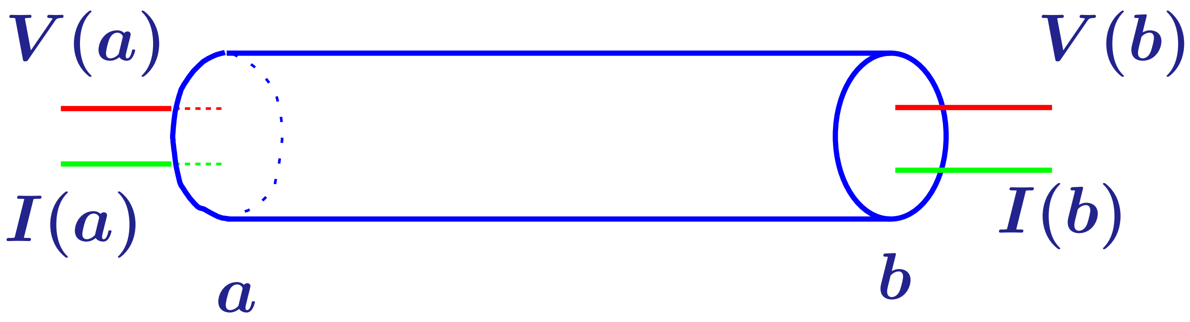

1
0.5
0
(z(t))(x)
2
1.5
1
0.5
0
t
1
0.5
0
x

- **The state is a function of the spatial variable.**
- **The mapping *initial state* to *state at time* t is a strongly continuous semigroup.**
- **There are conditions which tell you when the operator A generates a strongly continuous semigroup.**
- **The function $x(t) = T(t)x_0$ is always a (weak) solution of the abstract differential equation**

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

3 Linear Port Hamiltonian Systems

3.1 A typical example



Consider the transmission line on the spatial interval $[a, b]$

$$\frac{\partial Q}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta, t)}{L(\zeta)}$$
$$\frac{\partial \phi}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta}\frac{Q(\zeta, t)}{C(\zeta)}.$$

We write $x_1 = Q$ (charge) and $x_2 = \phi$ (flux), and we find that

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(\zeta,t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix} \frac{1}{C(\zeta)}x_1(\zeta,t) \\ \frac{1}{L(\zeta)}x_2(\zeta,t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix}\begin{pmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{pmatrix}\right]$$

$$= P_1\frac{\partial}{\partial \zeta}(\mathcal{H}x)(\zeta,t)$$

So we have, $J = P_1\frac{\partial}{\partial\zeta}$ (skew-symmetric) and the Hamiltonian (energy) equals $H = \frac{1}{2}\int_a^b x^T\mathcal{H}x d\zeta$ (positive).

To use this structure we differentiate the Hamiltonian (energy) along trajectories.

$$
\begin{aligned}
\frac{dH}{dt}(t) &= \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) dz + \\
&\quad \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) \frac{\partial x}{\partial t}(\zeta,t) dz \\
&= \frac{1}{2}\int_a^b \left(P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}x\right)(\zeta,t)\right)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta + \\
&\quad \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta,t) (P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}x\right)(\zeta,t) d\zeta \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial \zeta} \left[\left(\mathcal{H}x\right)^T (\zeta,t) P_1 \left(\mathcal{H}x\right)(\zeta,t) \right] d\zeta \\
&= \frac{1}{2} \left[\left(\mathcal{H}x\right)^T (\zeta,t) P_1 \left(\mathcal{H}x\right)(\zeta,t) \right]_a^b,
\end{aligned}
$$

where we have used the symmetry of P_1 and $\mathcal{H}$.

So we have that the time-change of Hamiltonian satisfies

$$\frac{dH}{dt}(t) = \frac{1}{2}\left[\left(\mathcal{H}x\right)^T(\zeta,t)P_1\left(\mathcal{H}x\right)(\zeta,t)\right]_a^b. \tag{1}$$

That is the change of internal energy goes via the boundary (ports).

Note the we only used that P_1 and $\mathcal{H}$ are symmetric. We did not need the specific form of P_1 or $\mathcal{H}$.

The balance equation (1) also holds for the system

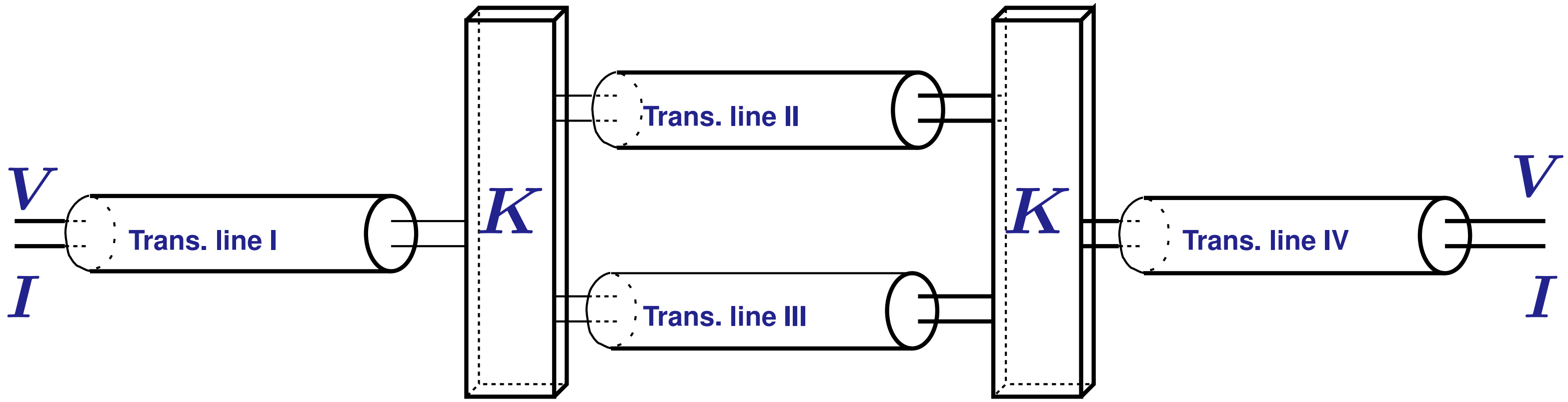
$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta,t) + P_0\left[\mathcal{H}x\right](\zeta,t) \tag{2}$$

with P_0 anti-symmetric, i.e., $P_0^T = -P_0$.

Many (hyperbolic) systems can be written in this format.

Example

Consider transmission lines in a network



In the coupling parts K, we have that Kirchhoff laws holds. Hence charge flowing out of the transmission line I, enters II and III, etc.

The P_1 of the big system is the diagonal matrix, build from the uncoupled P_1's (which are all the same). The $\mathcal{H}$ of the coupled system is the diagonal matrix of the uncoupled $\mathcal{H}$'s.

The coupling is written down as boundary conditions of the pde.

3.2 Homogeneous solutions of P.H.S.

Consider the p.d.e. of our Port-Hamiltonian system

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left[\mathcal{H}(\zeta) x(\zeta, t)\right] + P_0 \left[\mathcal{H}(\zeta) x(\zeta, t)\right].$$

with energy (Hamiltonian)

$$H(t) = \frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.$$

We assume that

- P_1 **is real and symmetric, i.e.,** $P_1^T = P_1$**, and** P_1 **is invertible,**
- P_0 **is real and anti-symmetric, i.e.,** $P_0^T = -P_0$**,**
- $\mathcal{H}(\zeta)$ **is a positive (real) symmetric matrix, uniformly satisfying** $0 < mI \leq \mathcal{H}(\zeta) \leq MI$**, almost everywhere.**

The energy

$$H(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta.$$

can be seen as a weighted L^2-norm.

Associated to this p.d.e. we define the state space

$$X = L^2((a,b);\mathbb{R}^n)$$

with inner product

$$\langle f, g\rangle = \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta) d\zeta.$$

and the differential operator

$$\mathfrak{A}x := P_1 \frac{d}{d\zeta}\left[\mathcal{H}x\right] + P_0\left[\mathcal{H}x\right]$$

with domain

$$D(\mathfrak{A}) = \{x \in X \mid x \text{ is absolutely continuous, and } \frac{dx}{d\zeta} \in X\}.$$

Thus the squared norm of x equals the energy of this state.

The following balance equation holds

Lemma

On the space X, the operator $\mathfrak{A}x = P_1 \frac{d}{d\zeta}[\mathcal{H}x] + P_0[\mathcal{H}x]$ satisfies

$$\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle$$
$$= \frac{1}{2}\left[(\mathcal{H}x)^T(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^T(a)P_1(\mathcal{H}x)(a)\right]$$

for $x \in D(\mathfrak{A})$. □

Proof (for $P_0 = 0$) Using the definition of the inner product, we find

$$\begin{aligned} &\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle \\ &= \frac{1}{2}\int_a^b \left[P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}x\right)(\zeta)\right]^T \mathcal{H}(\zeta)x(\zeta)d\zeta + \\ &\quad \frac{1}{2}\int_a^b x(\zeta)^T \mathcal{H}(\zeta)\left[P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}x\right)(\zeta)\right] d\zeta. \end{aligned}$$

Using the fact that $P_1, \mathcal{H}(\zeta)$ are real symmetric, we write the last

expression as

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[\frac{d}{d\zeta}\left(\mathcal{H}x\right)(\zeta)\right]^T P_1\mathcal{H}(\zeta)x(\zeta) + \\
&\qquad [\mathcal{H}(\zeta)x(\zeta)]^T \left[P_1\frac{d}{d\zeta}\left(\mathcal{H}x\right)(\zeta)\right] d\zeta \\
=&\frac{1}{2}\int_a^b \frac{d}{d\zeta}\left[\left(\mathcal{H}x\right)^T(\zeta)P_1\left(\mathcal{H}x\right)(\zeta)\right] d\zeta \\
=&\frac{1}{2}\left[\left(\mathcal{H}x\right)^T(b)P_1\left(\mathcal{H}x\right)(b) - \left(\mathcal{H}x\right)^T(a)P_1\left(\mathcal{H}x\right)(a)\right].
\end{aligned}$$

□

Theorem

Consider the operator A which is defined as

$$Ax = \mathfrak{A}x = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x) + P_0 (\mathcal{H}x)$$

on the domain $D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{B}$, where

$$\mathfrak{B}x = M \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix}.$$

If M is $n \times 2n$-matrix of full rank, then A generates a contraction semigroup on X if and only if

$$\langle Ax, x \rangle + \langle x, Ax \rangle \leq 0 \quad \text{for all } x \in D(A).$$

□

Hence by posing n boundary conditions, we only have to check the inequality.

Since we have the power balance this can be done quickly.

If $\mathcal{H}$ is non-constant, it can be very hard (impossible) to obtain the expression for the C_0-semigroup.

Example

Consider the transmission line.

$$\mathfrak{A}x = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{d}{d\zeta} \left[\begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix} \right]$$

and

$$\begin{aligned} &\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}\rangle \\ &\quad = \frac{1}{2}\left[(\mathcal{H}x)^T(b) P_1 (\mathcal{H}x)(b) - (\mathcal{H}x)^T(a) P_1 (\mathcal{H}x)(a) \right] \\ &\quad = V(a)I(a) - V(b)I(b), \end{aligned}$$

since $x_1/C = Q/C = V$ **and** $x_2/L = \phi/L = I$**.**

If we choose $V(a) = 0$ and $V(b) = RI(b)$, $R \geq 0$, then

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -R & 0 & 0 \end{bmatrix}.$$

This matrix has rank $n = 2$.

Furthermore, using these boundary conditions, we have that

$$V(a)I(a) - V(b)I(b) \leq 0.$$

Hence the operator A generates a contraction semigroup on Z. $\square$

4 Inputs and Outputs

4.1 Introduction

From the previous part we know what we mean by the solution of the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

However, what do we mean by the solution of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

To find the answer to this question is the aim of this part.

4.2 Inputs

In this section, we want to solve the abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

We assume that

- A **generates the** C_0**-semigroup** $(T(t))_{t\geq 0}$ **on the Hilbert space** X.
- $B \in \mathcal{L}(U, X)$.

So we want to find a (candidate) solution for

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

Choose $t_1 > 0$ and let $t \in [0, t_1]$.

We multiply the differential equation by $T(t_1 - t)$, and bring Ax to the left-hand side

$$T(t_1 - t)\dot{x}(t) - T(t_1 - t)Ax(t) = T(t_1 - t)Bu(t).$$

The left-hand side equals

$$\frac{d}{dt}\left[T(t_1 - t)x(t)\right] = T(t_1 - t)Bu(t).$$

Hence

$$\begin{aligned} \int_0^{t_1} T(t_1 - t)Bu(t)dt &= \left[T(t_1 - t)x(t)\right]_0^{t_1} \\ &= x(t_1) - T(t_1)x(0). \end{aligned}$$

Theorem

Consider the abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

where A generates the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, $B \in \mathcal{L}(U, X)$, and $u \in L^1_{\mathrm{loc}}((0,\infty);U)$. Then the (weak) solution is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau.$$

If u is continuously differentiable and $x_0 \in D(A)$, then it is the classical solution. □

Example

Consider the (controlled) p.d.e.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t) + u(t) \\ x(1, t) &= 0. \end{aligned}$$

We can write this as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with

$$Ax = \frac{dx}{d\zeta},$$

$$D(A) = \left\{ x \in L^2(0,1) \mid \frac{dx}{d\zeta} \in L^2(0,1) \text{ and } x(1) = 0 \right\}$$

and

$$Bu = 1 \cdot u.$$

Using the semigroup generated by A, we find that the solution of the p.d.e. is given by

$$x(\zeta, t) = x_0(\zeta + t)\, \mathbb{1}_{[0,1]}(\zeta + t) + \int_0^t \mathbb{1}_{[0,1]}(\zeta + t - \tau) u(\tau) d\tau.$$

with

$$\mathbb{1}_{[0,1]}(\xi) = \begin{cases} 1 & \xi \in [0,1] \\ 0 & \xi \notin [0,1] \end{cases}$$

□

Example

We consider the heated bar which is heated uniformly in the interval $[1/2, 1]$.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \mathbb{1}_{[1/2,1]}(\zeta)u(t) \\ \frac{\partial x}{\partial \zeta}(0, t) &= \frac{\partial x}{\partial \zeta}(0, t) = 0. \end{aligned}$$

This we can write as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with

$$(Bu)(\zeta) = \mathbb{1}_{[1/2,1]}(\zeta) \cdot u$$

and A as before

$$Ax = \frac{d^2x}{d\zeta^2},$$

with domain

$$D(A) = \left\{x \in L^2(0,1) \mid \frac{dx}{d\zeta} \text{ and } \frac{d^2x}{d\zeta^2} \in L^2(0,1), \frac{dx}{d\zeta}(0) = 0 = \frac{dx}{d\zeta}(1)\right\}.$$

□

4.3 Outputs

Now we have solved the input problem, the understanding of the output equation

$$y(t) = Cx(t) + Du(t)$$

is easy.

If $C \in \mathcal{L}(X, Y)$, and $D \in \mathcal{L}(U, Y)$, with Y a Hilbert space, then using the solution for $x(t)$ we find

$$
\begin{aligned}
y(t) &= C\left[T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau\right] + Du(t) \\
&= CT(t)x_0 + \int_0^t CT(t-\tau)Bu(\tau)d\tau + Du(t).
\end{aligned}
$$

Example

We take our heated bar, and we measure the (average) temperature in the other half of the bar.

$$
\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{2},1]}(\zeta)u(t) \\
\frac{\partial x}{\partial \zeta}(0,t) &= \frac{\partial x}{\partial \zeta}(0,t) = 0. \\
y(t) &= \int_0^{\frac{1}{2}} x(\zeta,t)d\zeta.
\end{aligned}
$$

This we can written as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

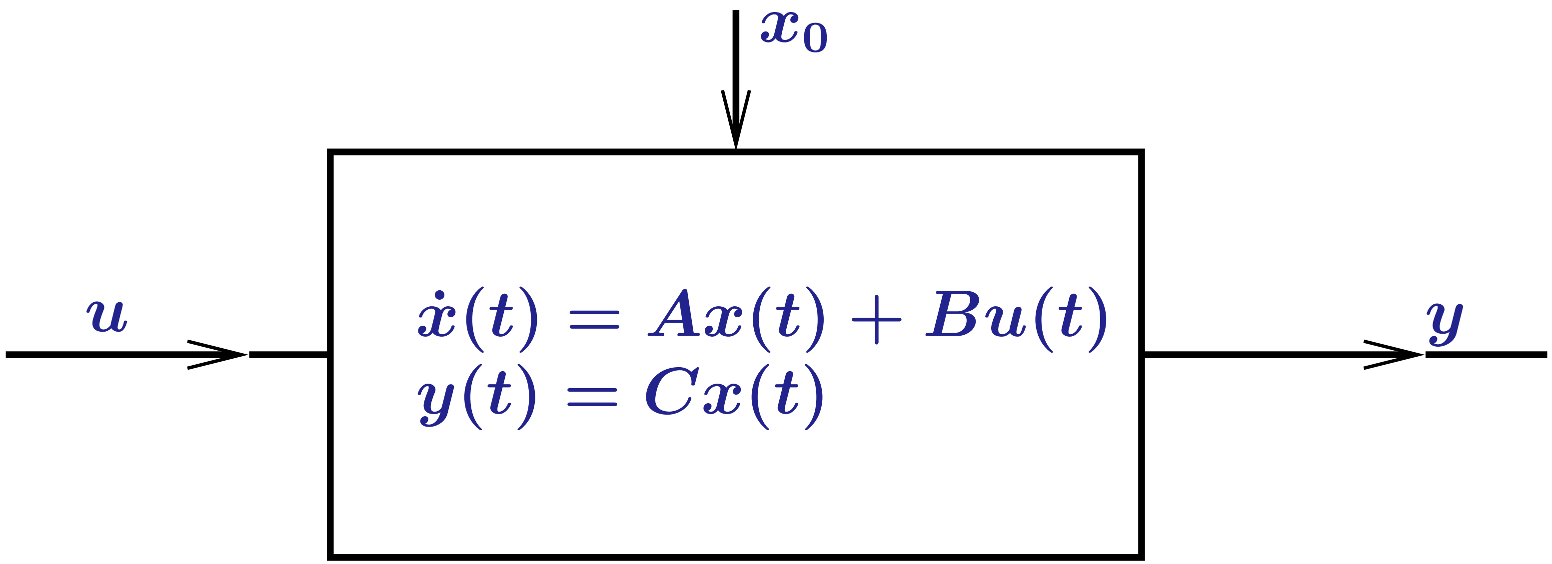
with A and B as before, and the operator $C \in \mathcal{L}(X, \mathbb{C})$ given by

$$
Cx = \int_0^{\frac{1}{2}} x(\zeta) d\zeta.
$$

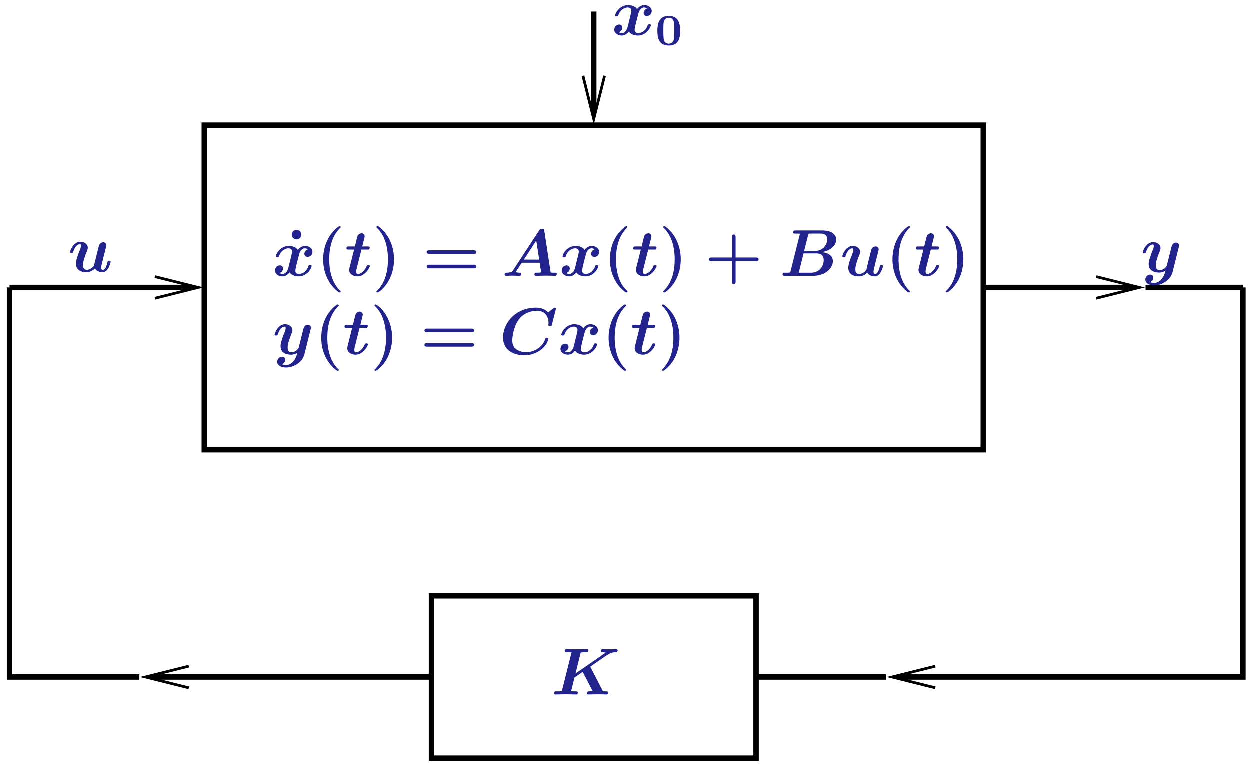
□

4.4 Feedback

We have defined the open loop system, schematically given by



However, for control we would like to close the loop.



This gives the differential equation

$$\dot{x}(t) = Ax(t) + BKCx(t) = (A + BKC)\, x(t).$$

Theorem

If A generates the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, and $B \in \mathcal{L}(U, X)$, $F \in \mathcal{L}(X, U)$, then $A + BF$ also generates a C_0-semigroup.

Furthermore, if we denotes this (closed-loop) semigroup by $(S(t))_{t\geq 0}$, then

$$S(t)x_0 = T(t)x_0 + \int_0^t T(t-\tau)BFS(\tau)x_0 d\tau.$$

□

4.5 Boundary control

Consider the following system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1] \\ x(1, t) &= u(t). \end{aligned}$$

We cannot write this in the form $\dot{x}(t) = Ax(t) + Bu(t)$, with $B \in \mathcal{L}(X, \mathbb{C})$.

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1]\\ x(1,t) &= u(t).\end{aligned}$$

We perform the following trick.

Define $v(\zeta,t) = x(\zeta,t) - u(t)$**.**

Then v **satisfies the following partial differential equation**

$$\begin{aligned}\frac{\partial v}{\partial t}(\zeta,t) &= \frac{\partial v}{\partial \zeta}(\zeta,t) - \dot{u}(t), \qquad \zeta \in [0,1]\\ v(1,t) &= 0.\end{aligned}$$

This we can write as $\dot{v}(t) = Av(t) + B\tilde{u}(t)$**, for** $\tilde{u} = \dot{u}$**.**

Definition

The system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0 \\ \mathfrak{B}x(t) &= u(t) \end{aligned}$$

is a boundary control system if

- **$\mathfrak{A}$ is a linear operator from $D(\mathfrak{A}) \subset X$ to X, $\mathfrak{B}$ is a linear operator from $D(\mathfrak{B}) \subset X$ to U, and $D(\mathfrak{A}) \subset D(\mathfrak{B})$.**
- **The operator A defined as $Ax = \mathfrak{A}x$ with domain $D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{B}$ generates a C_0-semigroup on X.**
- **The range of $\mathfrak{B}$ equals U.**

□

- Since $\mathfrak{B}$ is surjective, we can find a B such that for all $u \in U$
  - $Bu \in D(\mathfrak{B})$ and
  - $\mathfrak{B}Bu = u$.
- Define $v(t) = x(t) - Bu(t)$. Then
  - If $x \in D(\mathfrak{A})$, then $v \in D(\mathfrak{A})$, and
  - since $\mathfrak{B}v = 0$, we have that $v \in D(A)$.
- v satisfies the following abstract differential equation

$$
\begin{aligned}
\dot{v}(t) &= \mathfrak{A}x(t) - B\dot{u}(t) \\
&= \mathfrak{A}v(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\
&= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t).
\end{aligned}
$$

So if $x(t)$ is the classical solution of

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0 \\ \mathfrak{B}x(t) &= u(t), \end{aligned}$$

then $v(t) = x(t) - Bu(t)$ is a classical solution of

$$\begin{aligned} \dot{v}(t) &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ v(0) &= x_0 - Bu(0). \end{aligned}$$

The reserve direction also holds.

For this last differential equation:

$$\begin{aligned} \dot{v}(t) &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ v(0) &= x_0 - Bu(0) \end{aligned}$$

we know the solution

$$v(t) = T(t)v(0) + \int_0^t T(t-\tau)\left[\mathfrak{A}Bu(\tau) - B\dot{u}(\tau)\right] d\tau.$$

Example

We apply this to the transport equation with boundary control

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1] \\ x(1, t) &= u(t). \end{aligned}$$

Here we have that

$$\begin{aligned} \mathfrak{A}x &= \frac{dx}{d\zeta}, \qquad D(\mathfrak{A}) = \{x \in L^2(0,1) \mid \frac{dx}{d\zeta} \in L^2(0,1)\}. \\ \mathfrak{B}x &= x(1), \qquad D(\mathfrak{B}) = D(\mathfrak{A}). \end{aligned}$$

Let us check the assumptions.

$$\begin{aligned}\mathfrak{A}x &= \frac{dx}{d\zeta}, \qquad D(\mathfrak{A}) = \{x \in L^2(0,1) \mid \frac{dx}{d\zeta} \in L^2(0,1)\}.\\ \mathfrak{B}x &= x(1), \qquad D(\mathfrak{B}) = D(\mathfrak{A}).\end{aligned}$$

- **Both operators are linear.**
- $\mathfrak{A}$ **on the domain** $D(\mathfrak{A}) \cap \ker \mathfrak{B}$ **generates a** C_0**-semigroup.**
- **The range of** $\mathfrak{B}$ **is** $\mathbb{C} = U$**.**
- **Choose** $B = 1$**, then** $\mathfrak{B}Bu = u$**.**

Since $\mathfrak{A}B = 0$, we find

$$x(t) - 1 \cdot u(t) = v(t) = T(t)v_0 - \int_0^t T(t-\tau)1\dot{u}(\tau)d\tau.$$

Using the expression of the (shift) semigroup, we find that the solution of the boundary controlled p.d.e. is

$$\begin{aligned} x(\zeta,t) - u(t) &= v_0(\zeta+t)\,\mathbb{1}_{[0,1]}(\zeta+t) - \\ &\quad \int_0^t \mathbb{1}_{[0,1]}(\zeta+t-\tau)\dot{u}(\tau)d\tau. \end{aligned}$$

This we can simplify.

For $t > 1$ this expression becomes

$$\begin{aligned} x(\zeta, t) - u(t) &= v_0(\zeta + t)\, \mathbb{1}_{[0,1]}(\zeta + t) - \\ &\quad \int_0^t \mathbb{1}_{[0,1]}(\zeta + t - \tau)\dot{u}(\tau)d\tau \\ &= 0 - [u(\tau)]_{x+t-1}^{t} \\ &= -u(t) + u(\zeta + t - 1). \end{aligned}$$

For $t \in [0, 1]$ we have to distinguish two cases:

- $\zeta \in [0, 1-t]$:

$$\begin{aligned} x(\zeta, t) - u(t) &= v_0(\zeta + t)\, \mathbb{1}_{[0,1]}(\zeta + t) - \\ &\quad \int_0^t \mathbb{1}_{[0,1]}(\zeta + t - \tau)\dot{u}(\tau)d\tau \\ &= x_0(\zeta + t) - u(0) - [u(\tau)]_0^t \\ &= x_0(\zeta + t) - u(t). \end{aligned}$$

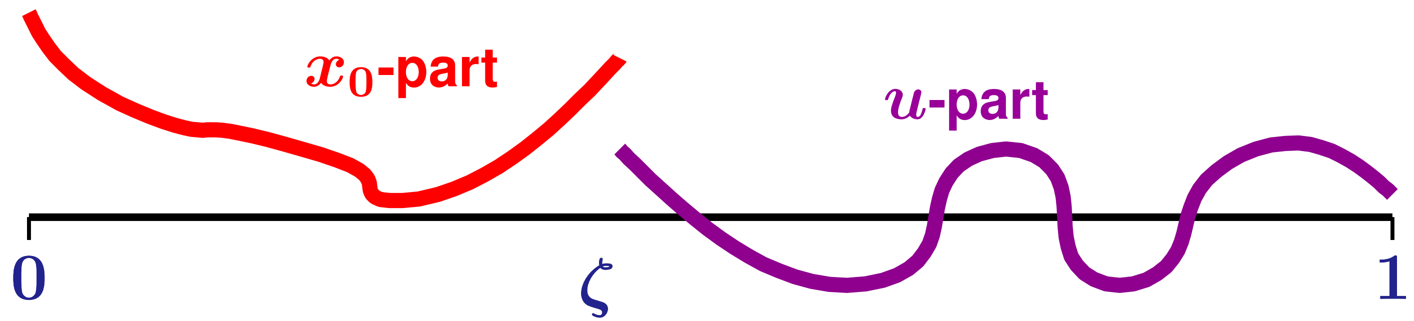
- $\zeta \in [1-t, 1]$:

$$\begin{aligned} x(\zeta, t) - u(t) &= 0 - [u(\tau)]_{\zeta+t-1}^t \\ &= -u(t) + u(\zeta + t - 1). \end{aligned}$$

This we can summarize as follows

$$x(\zeta,t) = \begin{cases} x_0(\zeta+t) & \zeta+t<1 \\ u(\zeta+t-1) & \zeta+t>1 \end{cases}$$

The solution for $t<1$ is depicted for all $\zeta \in [0,1]$.



Hence for the control transport equation we have found a (weak) solution for all L^2-input functions. □

4.6 Port-Hamiltonian boundary control systems

Consider the p.d.e. of our Port-Hamiltonian system

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left[\mathcal{H}(\zeta) x(\zeta, t)\right] + P_0 \left[\mathcal{H}(\zeta) x(\zeta, t)\right].$$

to which we add the boundary conditions (homogeneous and controlled)

$$u(t) = W_{B,1} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}$$

$$0 = W_{B,2} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}.$$

Theorem

Assume that the port-Hamiltonian system satisfies the usual conditions, assume further that the matrix $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ is a full rank $n \times 2n$ matrix, and for $u \equiv 0$ the (homogeneous) pde generates a contraction semigroup, then for general u the system is a boundary control system.

The above theorem implies that for smooth initial conditions x_0 and smooth inputs u satisfying

$$u(0) = W_{B,1} \begin{bmatrix} \mathcal{H}(b)x_0(b) \\ \mathcal{H}(a)x_0(a) \end{bmatrix},$$

the boundary output

$$y(t) = W_C \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}$$

is well-defined and continuous.

4.7 Summary

We have seen the following

- **A controlled p.d.e. like**

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \mathbb{1}_{[1/2,1]}(\zeta) u(t) \\ \frac{\partial x}{\partial \zeta}(0, t) &= \frac{\partial x}{\partial \zeta}(0, t) = 0 \\ y(t) &= \int_0^{\frac{1}{2}} x(\zeta, t) d\zeta \end{aligned}$$

can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with B, C, and D bounded operators.

- **We know what we mean by the solution of this (controlled) abstract differential equation, and we have a formula for it.**

- **If the control appears at the boundary, like**

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t) \\ x(1, t) &= u(t) \end{aligned}$$

we can write this in the abstract form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t) \\ \mathfrak{B}x(t) &= u(t) \end{aligned}$$

- **We know that we have a solution (for smooth inputs) of such a boundary control system.**
- **Port-Hamiltonian systems are boundary control systems.**

5 Stability and Stabilizability

5.1 Introduction

In this part we want to define stability and show how these notions relate to similar notions for ordinary differential equations.

Furthermore, we show how we can stabilize systems.

5.2 Stability notions

There are several stability notions for the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

We assume that A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X.

Definition

The semigroup $(T(t))_{t\geq 0}$ is exponentially stable if there exists $M, \omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}, \qquad t \geq 0.$$

□

Definition

The semigroup $(T(t))_{t\geq 0}$ is strongly stable if for all $x_0 \in X$

$$\lim_{t\to\infty} T(t)x_0 = 0.$$

□

5.3 Exponential stability and Lyapunov equations

The following theorem relates exponential stability with the solution of a Lyapunov equation.

Theorem

The following are equivalent.

- $(T(t))_{t\geq 0}$ **is exponentially stable.**
- **For all** $x_0 \in X$ **we have that** $T(t)x_0 \in L^2((0,\infty);X)$.
- **There exists a positive** $L \in \mathcal{L}(X)$ **satisfying the following Lyapunov equation**

$$\langle Ax, Lx\rangle + \langle Lx, Ax\rangle = -\langle x, x\rangle \qquad x \in D(A).$$

□

Some remarks concerning the Lyapunov equation.

Lemma

For a positive operator $L \in \mathcal{L}(X)$ the following are equivalent:

- **L satisfies**

$$\langle Ax, Lx\rangle + \langle Lx, Ax\rangle = -\langle x, x\rangle \qquad \forall x \in D(A).$$

- **L satisfies**

$$\langle Ax_1, Lx_2\rangle + \langle Lx_1, Ax_2\rangle = -\langle x_1, x_2\rangle \qquad \forall x_1, x_2 \in D(A).$$

- **L maps the domain of A into the domain of A^*, and L satisfies**

$$A^*L + LA = -I \qquad \text{on } D(A).$$

Proof of the equivalence between the last two items.

Assume that $x_0 \in L^2((0,\infty);X)$ for all $x_0 \in X$. For $x_1, x_2 \in X$, define

$$\langle x_1, Lx_2\rangle = \int_0^\infty \langle T(t)x_1, T(t)x_2\rangle dt.$$

This operator is well-defined, self-adjoint, and positive. Furthermore, for $x_1, x_2 \in D(A)$

$$\begin{aligned}\langle Ax_1, Lx_2\rangle + \langle x_1, LAx_2\rangle \\ = \int_0^\infty \langle T(t)Ax_1, T(t)x_2\rangle dt + \int_0^\infty \langle T(t)x_1, T(t)Ax_2\rangle dt.\end{aligned}$$

Hence

$$\begin{aligned}
&\langle Ax_1, Lx_2\rangle + \langle x_1, LAx_2\rangle \\
&\quad = \int_0^\infty \langle T(t)Ax_1, T(t)x_2\rangle dt + \int_0^\infty \langle T(t)x_1, T(t)Ax_2\rangle dt \\
&\quad = \int_0^\infty \frac{d}{dt}\left[\langle T(t)x_1, T(t)x_2\rangle\right] dt \\
&\quad = 0 - \langle x_1, x_2\rangle.
\end{aligned}$$

If the Lyapunov equation has a positive solution, then

$$
\begin{aligned}
&\int_0^{t_1} \|T(t)x_0\|^2 dt \\
&= \int_0^{t_1} \langle T(t)x_0, T(t)x_0\rangle dt = \int_0^{t_1} \langle T(t)x_0, IT(t)x_0\rangle dt \\
&= -\int_0^{t_1} \langle AT(t)x_0, LT(t)x_0\rangle dt \\
&\quad -\int_0^{t_1} \langle T(t)x_0, LAT(t)x_0\rangle dt \\
&= -\int_0^{t_1} \frac{d}{dt}\left[\langle T(t)x_0, LT(t)x_0\rangle\right] dt \\
&= -\langle T(t_1)x_0, LT(t_1)x_0\rangle + \langle x_0, Lx_0\rangle.
\end{aligned}
$$

Hence for all $t_1 > 0$

$$\int_0^{t_1} \|T(t)x_0\|^2 dt \leq \|L\| \|x_0\|^2$$

and so $T(t)x_0 \in L^2((0,\infty);X)$.

Thus we have proved the equivalence between the last two items.

Proof of the equivalence between the first two items.

If $T(t)x_0 \in L^2((0,\infty), X)$ for all x_0, then

- $\|T(t)\| \leq M$ for all $t > 0$
- The following estimate holds

$$\begin{aligned} t\|T(t)x_0\|^2 &= \int_0^t \|T(t)x_0\|^2 dt \\ &= \int_0^t \|T(t-\tau)T(\tau)x_0\|^2 d\tau \\ &\leq M^2 \int_0^t \|T(\tau)x_0\|^2 d\tau \leq K\|x_0\|^2. \end{aligned}$$

Thus

$$\|T(t)x_0\|^2 \leq \frac{K}{t}\|x_0\|^2.$$

Now using the semigroup property, we find that $(T(t))_{t\geq 0}$ is exponentially stable. $\square$

There are examples of unstable semigroups for which the infinitesimal generator A has no spectrum in the set $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$.

Hence, in general, we cannot conclude stability by only looking at the spectrum of A. However,

Theorem

The semigroup $(T(t))_{t\geq 0}$ is exponentially stable if and only if

$$\sup_{\{s\in\mathbb{C} \mid \mathrm{Re}(s)>0\}} \|(sI - A)^{-1}\| < \infty.$$

□

5.4 Strong stability

We may also conclude exponential stability when there exists a positive L such that

$$A^*L + LA \leq -\varepsilon I, \qquad \varepsilon > 0.$$

Question

Is the semigroup strongly stable when there exists an $L \in \mathcal{L}(X)$, $L > 0$ such that

$$A^*L + LA < 0 \qquad \text{on } D(A)?$$

Answer

No.

Example

Take the Hilbert space

$$Z = \left\{ f : [0,\infty) \mapsto \mathbb{C} \mid \int_0^\infty |f(x)|^2 \left[e^{-x} + 1\right] dx < \infty \right\}.$$

It inner product is given by:

$$\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)} \left[e^{-x} + 1\right] dx.$$

Note that Z "is" $L^2(0,\infty)$.

As semigroup, we choose the shift:

$$\begin{aligned} (T(t)f)(x) &= \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t) \end{cases} \\ &= f(x-t)\,\mathbb{1}_{[0,\infty]}(x-t). \end{aligned}$$

There holds

$$\begin{aligned}
\|T(t)f\|^2 &= \int_0^\infty |f(x-t)\,\mathbb{1}_{[0,\infty]}(x-t)|^2\left[e^{-x}+1\right]dx \\
&= \int_0^\infty |f(\xi)|^2\left[e^{-(\xi+t)}+1\right]d\xi \\
&\geq \int_0^\infty |f(\xi)|^2 d\xi \\
&\geq \frac{1}{2}\int_0^\infty |f(\xi)|^2\left[e^{-\xi}+1\right]d\xi \\
&= \frac{1}{2}\|f\|^2.
\end{aligned}$$

Hence $(T(t))_{t\geq 0}$ is not strongly stable.

The infinitesimal generator is given by

$$Af = -\frac{df}{dx}$$

with domain

$$D(A) = \left\{ f \in Z \mid f \text{ is absolutely continuous } \frac{df}{dx} \in Z, \text{ and } f(0) = 0 \right\}.$$

Next we evaluate

$$\langle Az, z \rangle + \langle z, Az \rangle$$

for $z \in D(A)$ and $z \neq 0$.

For $A = -\frac{d}{dx}$ with boundary condition $z(0) = 0$, we find

$$\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle & \\
= & \int_0^\infty (-1)\frac{dz}{dx}(x)\overline{z(x)}\left[e^{-x}+1\right]dx + \\
& \int_0^\infty z(x)(-1)\overline{\frac{dz}{dx}(x)}\left[e^{-x}+1\right]dx \\
= & -\int_0^\infty \frac{d}{dx}\left(|z(x)|^2\right)\left[e^{-x}+1\right]dx.
\end{aligned}$$

Hence

$$
\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle \\
&= -\left[|z(x)|^2\left[e^{-x}+1\right]\right]_0^{\infty} + \\
&\quad \int_0^{\infty} |z(x)|^2\left[-e^{-x}\right] dx \\
&= 0 - \int_0^{\infty} |z(x)|^2 e^{-x} dx \\
&< 0.
\end{aligned}
$$

Concluding, we have constructed an infinitesimal generator A for which the semigroup is not strongly stable, but the Lyapunov inequality

$$\langle Az, z\rangle + \langle z, Az\rangle < 0, \qquad z \in D(A), z \neq 0$$

holds. Note that $L = I$. □

Remark

If the trajectories, $t \mapsto T(t)x_0$, are pre-compact, then this Lyapunov inequality implies strong stability. □

Theorem

Let $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on the Hilbert space Z. If

- **The semigroup is uniformly bounded, i.e., $\|T(t)\| \leq M$**
- **A has no eigenvalues on the imaginary axis**
- **The spectrum on the imaginary axis is countable,**

then $(T(t))_{t\geq 0}$ is strongly stable.

5.5 Stabilizability

In this section we study the question whether there exist a feedback, $u = Fx$ which stabilizes the system

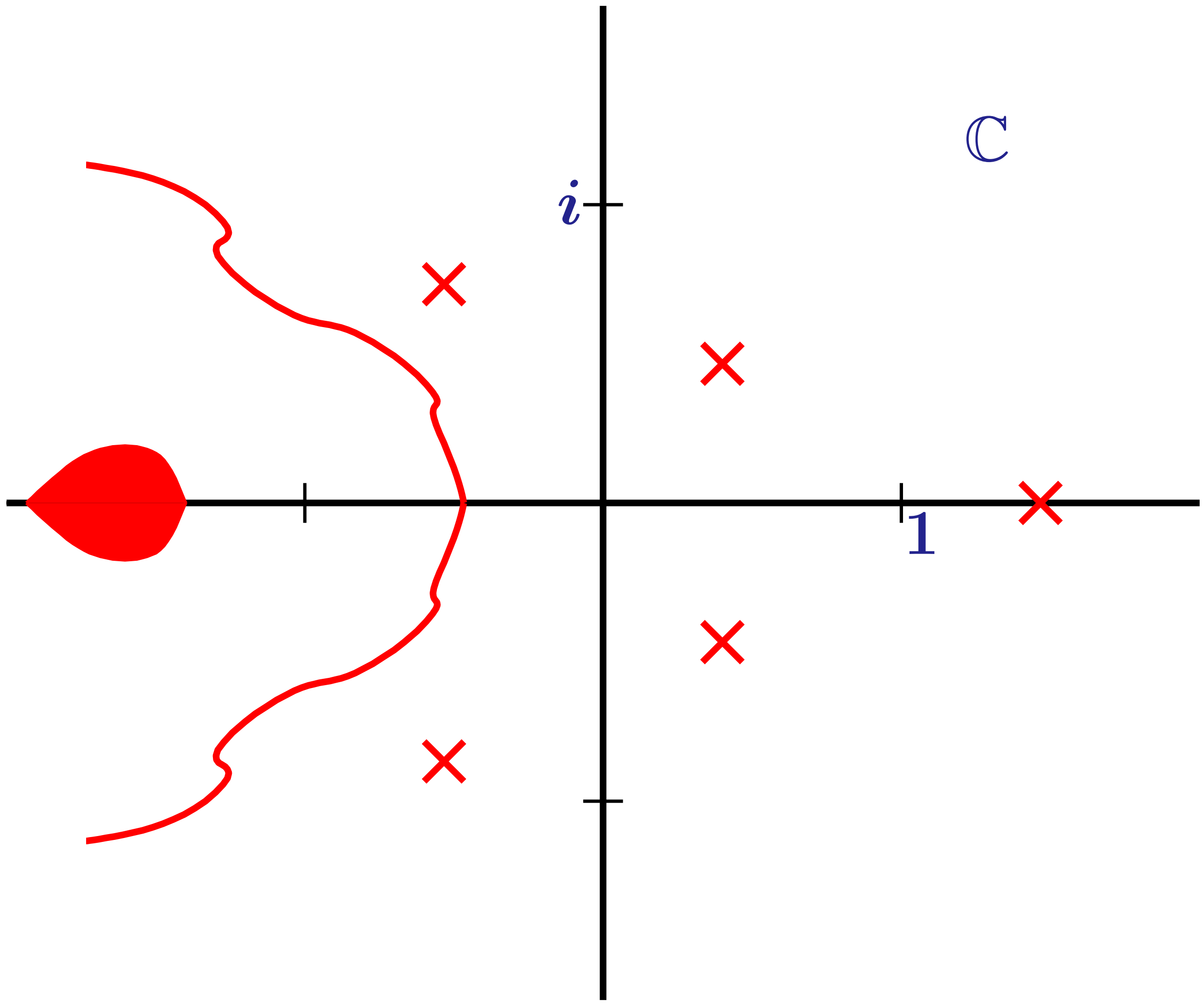
$$\dot{x}(t) = Ax(t) + Bu(t).$$

That is, the operator $A + BF$ generates an exponentially stable semigroup.

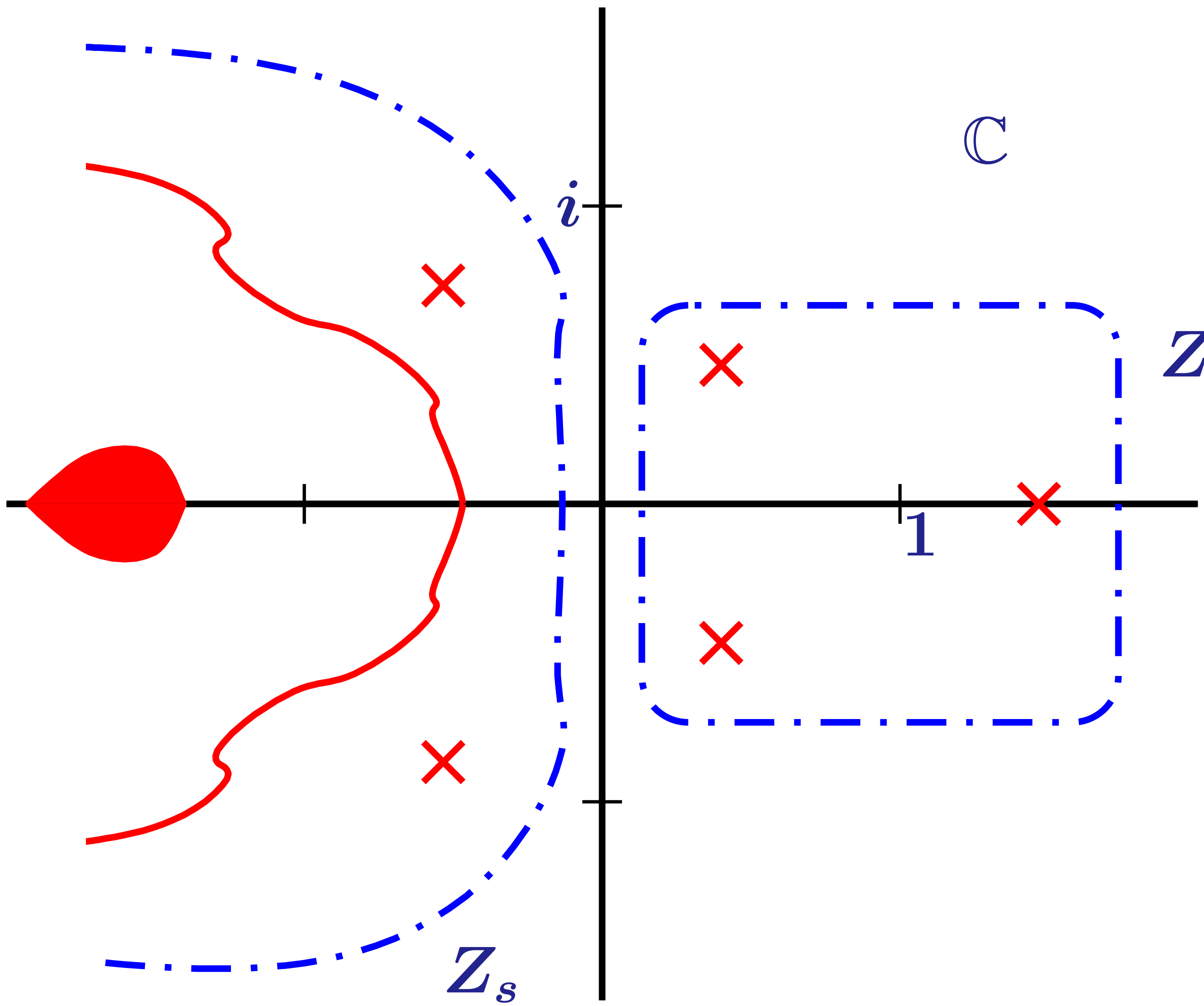
Theorem

Consider $\dot{x}(t) = Ax(t) + Bu(t)$, with $B \in \mathcal{L}(\mathbb{C}^m, X)$. The following are equivalent.

- **There exists an $F \in \mathcal{L}(X, \mathbb{C}^m)$ such that $A + BF$ generates an exponentially stable semigroup on X.**
- **Z can be decomposed as $X = X_u \oplus X_s$ with**
  - $\dim(X_u) < \infty$.
  - **For all $t > 0$ we have that $T(t)X_u \subset X_u$ and $T(t)X_s \subset X_s$.**
  - **$T(t)|_{X_s}$ is exponentially stable.**
  - **The system restricted to X_u is controllable.**



Red denotes the spectrum.


ℂ
i
1
Z_u
Z_s

Example

We consider the heated bar. We heat it uniformly at one half, and we measure (half) the average temperature in the other half.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \mathbb{1}_{[\frac{1}{2},1]}(\zeta)u(t) \\ \frac{\partial x}{\partial \zeta}(0, t) &= \frac{\partial x}{\partial \zeta}(1, t) = 0 \\ y(t) &= \int_0^{\frac{1}{2}} z(\zeta, t)d\zeta. \end{aligned}$$

We know that the corresponding A has the eigenvalues

$\lambda_n = -n^2\pi^2, n = 0, 1, 2, \ldots$, and the eigenfunctions

$$\phi_n(\zeta) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\cos(n\pi\zeta) & n = 1, 2, \ldots \end{cases}$$

This A has an unstable eigenvalue, $\lambda_0 = 0$. Now we define

$$X_u = \text{span}\{\phi_0\}$$

$$X_s = \text{span}_{n=1,2,\ldots}\{\phi_n\} = X_u^{\perp}.$$

Let us check the conditions.

- $\dim(X_u) = 1 < \infty$.
- $T(t)X_u \subset X_u$ and $T(t)X_s \subset X_s$.
-
$$T(t)|_{X_s} = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle \cdot, \phi_n\rangle \phi_n.$$
  This is exponentially stable.
- A restricted to X_u is $A_u = [0]$, and the projection of B onto X_u is $B_u = \langle B, \phi_0\rangle = [1/2]$.
  The pair (A_u, B_u) is controllable.

Hence the system is stabilizable.

You can stabilize it by stabilizing the finite-dimensional part, and not effecting the stable part.

A feedback that works is

$$Fx = -k\langle x, \phi_0 \rangle \phi_0,$$

with $k > 0$. □

5.6 Summary

- **We have introduce different notions of stability for a C_0-semigroup.**
- **For exponential stability there are nice equivalent conditions.**
- **For strong stability there are nice sufficient conditions, but they are not sufficient.**
- **If the input is finite-dimensional, then there is a complete characterizing of stabilizability.**

6 Linear Quadratic Optimal Control

The linear quadratic optimal control problem (on an infinite horizon) is to minimize the following cost criterium w.r.t. u

$$J(x_0, u) = \int_0^\infty \|y(t)\|^2 + \langle u(t), Ru(t)\rangle dt,$$

where y **is the solution of the state linear system** $\Sigma(A, B, C)$ **with input** u **and initial condition** x_0**, i.e**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

and $R = R^* \in \mathcal{L}(U)$ **with** $R \geq \varepsilon I, \varepsilon > 0$.

Theorem

For the system $\Sigma(A, B, C)$ with cost criterion $J(x_0, u)$ the following are equivalent.

- **The system is optimizable, i.e., for every $x_0 \in X$ there exists an input u such that $J(x_0, u) < \infty$.**
- **For every $x_0 \in X$ there exists an (unique) input u^{opt} such that $J(x_0, u^{\text{opt}}) \leq J(x_0, u)$ for all u.**
- **There exists a non-negative solution to the ARE:**

  $P \in \mathcal{L}(X)$, $P \geq 0$ and

$$A^*P + PA - PBR^{-1}B^*P + C^*C = 0.$$

Hence to solve the optimal control problem, you only have to show that the system is optimizable. This can be very hard, but there are sufficient conditions.

- **If the pair (A, B) is stabilizable, then the system is optimizable.**
- **Furthermore, the smallest non-negative solution P of the ARE gives the optimal solution, i.e., $u^{\mathrm{opt}} = -R^{-1}B^*Px$.**
- **If, additionally, the pair (C, A) is detectable (there exists a $G \in \mathcal{L}(Y, X)$ such that $A + GC$ is exponentially stbale), then the non-negative solution of the ARE is unique, and the optimal feedback $-R^{-1}B^*P$ is exponentially stabilizable.**

Remark

Note the following link between the ARE and the Lyapunov equation:

The ARE can be written as

$$
\begin{aligned}
0 &= A^*P + PA - PBR^{-1}B^*P + C^*C \\
&= (A - BR^{-1}B^*P)^*P + P(A - BR^{-1}B^*P) + \\
&\quad PBR^{-1}B^*P + C^*C.
\end{aligned}
$$

Hence with $A_P := A - BR^{-1}B^*P$ we have that the ARE equals the Lyapunov equation

$$A_P^*P + PA_P = -PBR^{-1}B^*P - C^*C.$$

7 Semi-linear Differential Equations

In this part we study abstract differential equations of the form

$$\dot{x}(t) = Ax(t) + f(x(t)), \qquad x(0) = x_0,$$

where A generates the C_0-semigroup $T(t)$ on the Hilbert space X and f is a non-linear term.

Since we assume that Ax is "more important" than $f(x)$, we call these equations semi-linear.

7.1 Existence of solutions

Consider the semi-linear equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \qquad x(0) = x_0,$$

where A generates the C_0-semigroup $T(t)$ on the Hilbert space X and f is a Lipschitz continuous function from X to X, i.e.,

$$\|f(x_1) - f(x_2)\| \leq L_r \|x_1 - x_2\| \text{ for } \|x_1\|, \|x_2\| \leq r.$$

Then the following theorem holds:

Theorem

For every $x_0 \in X$ there exists a time $t_{\max} > 0$ such that a unique (continuous) solution exists on $[0, t_{\max})$.

When $t_{\max} < \infty$, then $\lim_{t \uparrow t_{\max}} \|x(t)\| = \infty$.

Remark

The solution satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(x(\tau))d\tau.$$

In fact, the proof shows that the above mapping has a fixed point.

The condition that f is Lipschitz continuous from X to X is very strong. For instance, for $X = L^2(0,1)$ the following functions do not satisfy it;

- $f(x) = \frac{dx}{d\zeta}x$,
- $f(x) = x^3$.

The famous viscous Burgers equation has the above non-linear term,

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \varepsilon\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \frac{\partial x}{\partial \zeta}(\zeta,t)x(\zeta,t) \\ x(0,t) &= x(1,t) = 0. \end{aligned}$$

Idea: Assume that $-A$ is non-negative, so it has a square root $(-A)^{\frac{1}{2}}$, then

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + f(x(t)) \iff \\
\dot{x}(t) &= Ax(t) + f((-A)^{-\frac{1}{2}}(-A)^{\frac{1}{2}}x(t)) \iff \\
(-A)^{\frac{1}{2}}\dot{x}(t) &= A(-A)^{\frac{1}{2}}x(t) + (-A)^{\frac{1}{2}}f((-A)^{-\frac{1}{2}}(-A)^{\frac{1}{2}}x(t)) \\
& \text{choosing } v(t) = (-A)^{\frac{1}{2}}x(t) \text{ gives} \\
\dot{v}(t) &= Av(t) + (-A)^{\frac{1}{2}}f((-A)^{-\frac{1}{2}}v(t)).
\end{aligned}$$

Assume now that $\tilde{f}(v) := f((-A)^{-\frac{1}{2}}v)$ is Lipschitz continuous on X, then we have to give meaning to

$$v(t) = T(t)v_0 + \int_0^t T(t-\tau)(-A)^{\frac{1}{2}}\tilde{f}(v(\tau))d\tau$$

Idea, continued: To give a meaning to the convolution

$$\int_0^t T(t-\tau)(-A)^{\frac{1}{2}}\tilde{f}(v(\tau))d\tau,$$

we consider the term $T(t-\tau)(-A)^{\frac{1}{2}}$.

Lemma

If $-A$ is non-negative, then $\|T(t-\tau)(-A)^{\frac{1}{2}}\| \leq M/\sqrt{t-\tau}$.

Hence the convolution can be given a meaning.

Remark

The above lemma holds for many parabolic pde's, even for those in which $-A$ is not non-negative.

Theorem

Consider the semi-linear equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \qquad x(0) = x_0,$$

where A generates the C_0-semigroup $T(t)$ on the Hilbert space X, $-A \geq 0$, and $f((-A)^{-\frac{1}{2}}v)$ is a Lipschitz continuous function from X to X, or equivalently

$$\|f(x_1) - f(x_2)\| \leq L_r \|(-A)^{\frac{1}{2}}x_1 - (-A)^{\frac{1}{2}}x_2\|$$

for $\|(-A)^{\frac{1}{2}}x_1\|, \|(-A)^{\frac{1}{2}}x_2\| \leq r$.

Then for every $x_0 \in D((-A)^{\frac{1}{2}})$ there exists a time $t_{\max} > 0$ such that a unique (continuous) solution exists on $[0, t_{\max})$. When $t_{\max} < \infty$, then $\lim_{t\uparrow\ t_{\max}} \|(-A)^{\frac{1}{2}}x(t)\| = \infty$.

In general it can be hard to find $(-A)^{\frac{1}{2}}$. However, its domain and norm often can be found.

Example

For

$$A = \frac{d^2}{d\zeta^2},$$

with domain

$$D(A) = \{f \in L^2(0,1) \mid \frac{d^2 f}{d\zeta^2} \in L^2(0,1), f(0) = f(1) = 0\}$$

we have

$$D((-A)^{\frac{1}{2}}) = \{f \in L^2(0,1) \mid \frac{df}{d\zeta} \in L^2(0,1), f(0) = f(1) = 0\}$$

and

$$\|(-A)^{\frac{1}{2}} f\| = \|\frac{df}{d\zeta}\|$$

Remark

Note that

- $(-A)^{\frac{1}{2}}f \neq \frac{df}{d\zeta}$
- $\{f \in L^2(0,1) \mid \frac{df}{d\zeta} \in L^2(0,1), f(0) = f(1) = 0\} =: H_0^1(0,1)$

As a consequence of our theorem, we have that the viscous Burgers equation;

$$
\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \varepsilon\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t)+\frac{\partial x}{\partial \zeta}(\zeta,t)x(\zeta,t)\\
x(0,t) &= x(1,t)=0
\end{aligned}
$$

possesses a unique solution for every $x_0 \in H_0^1(0,1)$.

7.2 Positivity

Definition

We call the function $f \in L^2(a,b)$ positive, when $f(\zeta) \geq 0$ for almost all $\zeta \in (a,b)$. Notation: $f \geq 0$

We call the semi-linear abstract differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \qquad x(0) = x_0$$

with state space $X \subset L^2(a,b)$ positive if $x(t) \geq 0$ for all $x_0 \geq 0$ and $t \in (0, t_{\max})$.

We show positivity for our pde.

Example

Consider the pde for $\zeta \in (0,1)$ and $t \geq 0$

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \varepsilon \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \frac{\partial x}{\partial \zeta}(\zeta,t) x(\zeta,t) \\ x(0,t) &= x(1,t) = 0 \\ x(\zeta,0) &= x_0(\zeta) \end{aligned}$$

Idea: If $x(\zeta, t)$ is negative for some t and ζ, then $x(\zeta, t)$ has a (negative) minimum on the compact set $[0, 1] \times [0, t_{\text{end}}]$. So there exists ζ_1, t_1 such that

$$\frac{\partial x}{\partial t}(\zeta_1, t_1) = 0 \quad \text{and} \quad \frac{\partial x}{\partial \zeta}(\zeta_1, t_1) = 0.$$

Furthermore,

$$\frac{\partial^2 x}{\partial \zeta^2}(\zeta_1, t_1) > 0.$$

The pde

$$\frac{\partial x}{\partial t}(\zeta, t) = \varepsilon \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \frac{\partial x}{\partial \zeta}(\zeta, t) x(\zeta, t)$$

contradicts this.

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