Day I : distributed parameters systems modelling - the classical approach

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School: *Introduction to modelling and control of systems governed by PDEs*
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Outline

An introduction to distributed parameters systems

Complex water systems

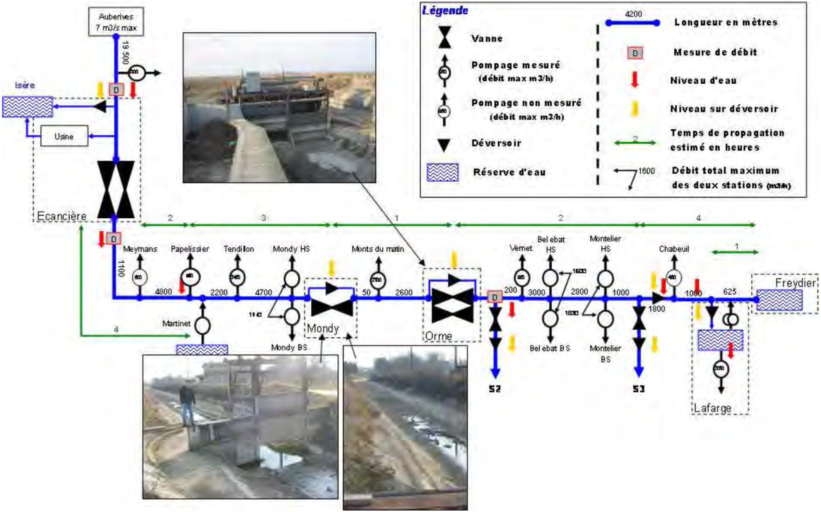
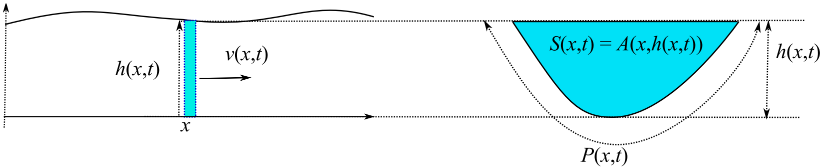


Figure: Schematic view of the Bourne irrigation system near Valence

the shallow water equations (SWE)



State (energy) variables are:

$$
\begin{aligned}
q(x,t) &= \rho S(x,t) && \textit{mass density} \\
p(x,t) &= \rho v(x,t) && \textit{momentum density}
\end{aligned}
$$

Conservation equations

The simplified Saint-Venant equations is a system of 2 conservation laws

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= -\frac{\partial}{\partial x}\left(S(x,t)v(x,t)\right) && \textbf{mass} \\
\frac{\partial p}{\partial t} &= -\frac{\partial}{\partial x}\left(\rho\left(gh(x,t)+\frac{v^2(x,t)}{2}\right)\right) && \textbf{momentum}
\end{aligned}
$$

Shallow water equations: a control system

Example of control system for the SWE (single reach)

$$\begin{aligned} \frac{\partial q}{\partial t} &= -\frac{\partial}{\partial x}\left(S(x,t)v(x,t)\right) + w(x,t) \\ \frac{\partial p}{\partial t} &= -\frac{\partial}{\partial x}\left(\rho\left(gh(x,t) + \frac{v^2(x,t)}{2}\right)\right) + kw(x,t)v(x,t) \end{aligned}$$

Boundary conditions (sliding gates):

$$\begin{aligned} Q(0,t) &:= Sv|_{x=0} = \alpha u_1(t)\sqrt{h_{in}(t) - h(0,t)} \\ Q(L,t) &:= Sv|_{x=L} = \alpha u_2(t)\sqrt{h(L,t) - h_{out}(t)} \end{aligned}$$

where

- $w(x,t)$, $h_{in}(t)$, $h_{out}(t)$ are external variables (withdrawals, water levels)
- $(Q(0,t), h(0,t))$, $(Q(L,t), h(L,t))$ are interconnection variables
- $u_1(t)$, $u_2(t)$ are controlled variables (gates openings) or "inputs"
- $h(0,t))$, $h(L,t))$ are measured variables or "outputs"

The abstract differential problem for control systems

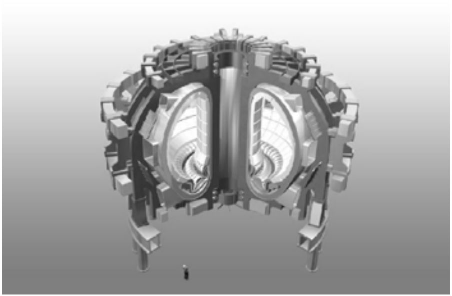
Abstract (partial) differential problem

$$
\begin{aligned}
\frac{\partial z}{\partial t}(t,x) = \mathcal{L}_x\left(z,t\right)(t,x) \quad &; \quad x \in \Omega, t > 0 \\
\mathcal{B}_x\left(z\right)(x,t) = 0 \quad &; \quad x \in \partial\Omega, t > 0 \\
z(0,x) = z_0(x) \quad &; \quad x \in \Omega
\end{aligned}
$$

where

- Ω is the **spatial domain** with **boundary** $\partial\Omega$
- $z(t,x)$ is the state and $z_0(x)$ the **initial state profile**
- $\mathcal{L}_x\left(z,t\right)$ and $\mathcal{B}_x\left(z,t\right)$ are **non autonomous differential operator** acting on the space variable x

Tokamak fusion reactors



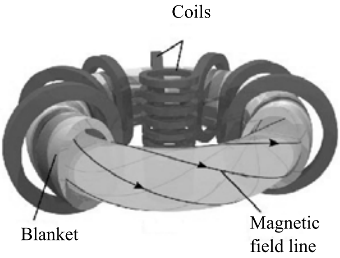


Figure: Schematic view of the coming ITER or existing Tore Supra tokamak reactors at CEA - Cadarache (Saint-Paul lez Durance)

the resistive diffusion equation

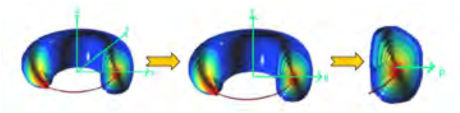


Figure: reduction asumptions

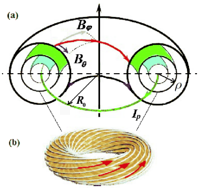


Figure: toric magnetic coordinates

The state variable is the poloidal plasma magnetic flux $\psi(\rho, t)$

MHD modelling and the resistive diffusion equation

$$\frac{\partial \psi}{\partial t} = \frac{\eta}{\rho} \frac{\partial}{\partial \rho} \left(\frac{1}{\mu} \rho \frac{\partial \psi}{\partial \rho} \right)$$

with η, the resistivity, μ, the permitivity and ρ the magnetic surface index or reduced radial coordinate.

Other examples of distributed parameters systems

- academic examples: wave, heat, beam, membrane and telegrapher equations
- piezzoelectric actuators
- fluid flow and fluid/structure control systems
- magneto-hydrodynamic flows, plasma control
- classical Maxwell field equations
- quantum mechanics (Schrödinger and Klein-Gordon equations)
- free surface problems (Burger, Korteweg de Vries, Boussinesq, Saint-Venant/SWE)
- thermodynamics (chemical reactions, transport phenomena, phases equilibrium, etc.)
- biology (preys-predators, population dynamics, bio-reactors)

$\Rightarrow$ most not simplified real world applications lead to DPS models!

Some characteristics of distributed parameters systems

- variables are **non uniform** in space, state equations are **Partial Differential Equations** (PDEs)
- **Boundary Control Systems** or **distributed control**
- extensions of classical control results exist for linear DPS using **semigroup theory**: transfer functions, I/O operators, controllability or observability Grammians, state space realization, LQG or H_∞ control, ...
- results exist for the **regional analysis** and control of linear (or bilinear) DPS (regional controllability or observability, spreadability, viability, etc.)

- practical solutions for control laws or observers design remain hard to achieve (solution of operators equations, infinite dimensional control)
- they are no general results for nonlinear DPS

$\Rightarrow$ *Particular cases*: **systems of conservation laws** with I/O or port variables

Different approaches for the control of DPS

- finite dimensional approximation (total or partial discretization)

  [Quarteroni et Valli 1994] *Numerical Approximation of Partial Differential Equations*, Springer-Verlag

  [Zuazua 2002] *Controllability of partial differential equations and its semi- discrete approximations*, Disc Cont Dyn Syst 8 (2) 469-513
- discrete modelling approaches (cellular automata, Lattice Boltzmann models)

  [Chopard et Droz 1998] *Cellular Automata Modeling of Physical Systems*, Cambridge University Press
- semigroup of linear operators (extensions to the semilinear case)

  [Curtain et Zwart 1995] *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag
- PDE / functional analysis approach

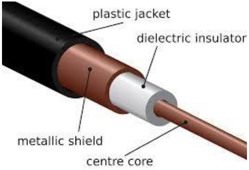
  [Lions 1988] *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Rev.30 pp. 1-68
- regional analysis (mainly using PDE/semigroup approaches)

  [El Jai et al. 2009] *Systèmes dynamiques : Analyse régionale des systèmes linéaires distribués*, PUP
- port-Hamiltonian approach

  [Duindam et al. 2009] *Modeling and control of complex physical systems: the Port-Hamiltonian approach*, Springer-Verlag

Simple examples and basic ideas about the dynamics and boundary conditions in DPS

Hyperbolic example (1/2)

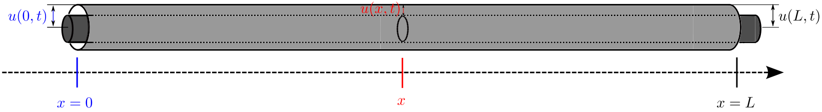


The **telegrapher's equation**

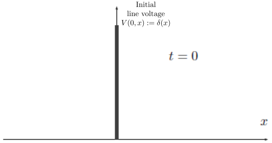
$$\frac{\partial^2 V}{\partial t^2}(t,x) = \frac{1}{\gamma l}\frac{\partial^2 V}{\partial x^2}(t,x) - \frac{r}{l}\frac{\partial V}{\partial t}(t,x)$$

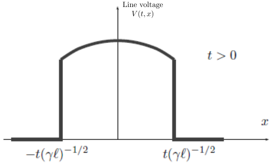
with

- γ, the lineic capacitance (farads per unit length)
- l, the lineic inductance (henries per unit length)
- r, the lineic resistance (ohms per unit length)
- $\dfrac{1}{\sqrt{\gamma l}}$, the propagation speed



Hyperbolic example (2/2)





$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{\gamma l}\frac{\partial^2 V}{\partial x^2} - \frac{r}{l}\frac{\partial V}{\partial t}$$

The solution with initial impulse in $x = 0$ (unbounded spatial domain):

$$V(t,x) = e^{\frac{-rt}{2l}} I_0\left(\frac{r}{2l}\sqrt{t^2 - \gamma l x^2}\right)$$

where I_0 is some modified Bessel's function [Courant & Hilbert, vol. 2]

- compact support
- **finite propagation speed** $c =\in \sqrt{\gamma l}$
- **shocks** (no regularization)

Parabolic example (1/2)

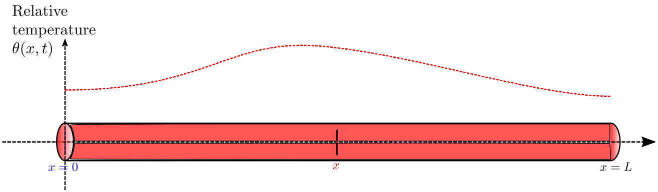
Jean Baptiste Joseph Fourier (1768-1830)

The **heat equation**

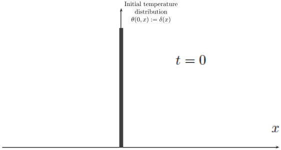
$$\frac{\partial \theta}{\partial t}(t,x) = \frac{D_t}{\rho C_p}\frac{\partial^2 \theta}{\partial x^2}(t,x)$$

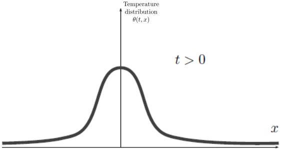
with

- D_t, the thermal conductivity $[J\, s^{-1} m^{-1} K^{-1}]$
- ρ, the lineic mass $[kg\, m^{-1}]$
- C_p, the specific heat capacity $[J\, kg^{-1} K^{-1}]$
- $k := \frac{D_t}{\rho C_p}$, the thermal diffusivity $[m^2 s^{-1}]$



Parabolic example (2/2)





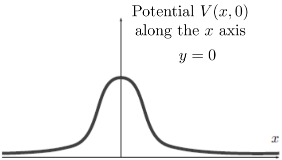
$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$$

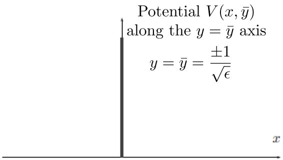
The solution with initial impulse in $x = 0$ (unbounded spatial domain):

$$\theta(t,x) = \sqrt{\frac{k}{4\pi t}} e^{-\frac{kx^2}{4t}}$$

- unbounded support for the solution ($t > 0$)
- **infinite propagation speed $c = \in \sqrt{\gamma l}$**
- **no shocks** (regularization/damping)

Elliptic example





The Laplace equation for the potential

$$\frac{\partial^2 V}{\partial y^2}(y,x)+\frac{\partial^2 V}{\partial x^2}(y,x)=0$$

with “initial” condition

$$V(0,x)=\frac{1}{\epsilon+x^2}\ \forall x\in\mathbb{R}$$

has solution

$$V(y,x)=\mathrm{Re}\left(\frac{1}{\epsilon+(x+iy)^2}\right)$$

- singularities may arise in the y direction from smooth profiles ($V(x,0)$)

Pierre Simon Laplace
(1749-1827)

Classification of second order PDEs (1/2)

Let us consider a PDE of the form

$$a\frac{\partial^2 z}{\partial x^2}(x,y) + 2b\frac{\partial^2 z}{\partial x \partial y}(x,y) + c\frac{\partial^2 z}{\partial y^2}(x,y) + d = 0$$

where a, b, c, d may depend on $x, y, z, \frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Then using a change of variables

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The PDE reduces to

$$A\frac{\partial^2 z}{\partial x^2}(x,y) + C\frac{\partial^2 z}{\partial y^2}(x,y) + D = 0$$

where A, C, D may depend on $X, Y, z, \frac{\partial z}{\partial X}$ and $\frac{\partial z}{\partial Y}$.

Classification of second order PDEs (2/2)

- **hyperbolic form**: $AC < 0 \Leftrightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} < 0$

$$\text{Wave equation: } \frac{\partial^2 z}{\partial t^2}(t,x) - c^2 \frac{\partial^2 z}{\partial x^2}(t,x) = 0$$

- **elliptic form** $AC > 0 \Leftrightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} > 0$

$$\text{Laplace equation: } \frac{\partial^2 z}{\partial x^2}(x,y) + \frac{\partial^2 z}{\partial y^2}(y,x) = 0$$

- **parabolic form** $AC = 0 \Leftrightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\text{Heat equation: } \mathbf{0}\frac{\partial^2 z}{\partial t^2} + k\frac{\partial^2 z}{\partial x^2}(t,x) - \frac{\partial z}{\partial t}(t,x) = 0$$

Remark: Same ideas apply for second order PDEs with N independent variables

Boundary conditions, the wave equation example (1/5)

Jean Le Rond D'Alembert (1717-1783)

1D wave equation - unbounded spatial domain $x \in \mathbb{R}$

state equation: $$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

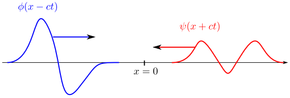
initial conditions: $$z(x,0) = z_0(x) \quad \forall x \in \mathbb{R}$$ $$\frac{\partial z}{\partial t}(x,0) = z_1(x) \quad \forall x \in \mathbb{R}$$

where z_0 and z_1 are smooth position and velocity profiles

D'Alembert general solution - travelling waves

$$z(x,t) = \phi(x - ct) + \psi(x + ct)$$

where $\phi(\cdot)$ and $\psi(\cdot)$ arbitrary smooth functions



Boundary conditions, the wave equation example (2/5)

With position and velocity **initial conditions**

$$\begin{cases} z_0(x) = \phi(x) + \psi(x) \\ z_1(x) = -c\phi'(x) + c\psi'(x) \end{cases}$$

one gets **waveform solutions**

$$\begin{cases} \phi(x) = z_0(x) - \psi(x) \\ \psi(x) = \int_0^x \frac{1}{2c} \left[z_1(\xi) + c\frac{dz_0}{d\xi}(\xi) \right] d\xi + A \end{cases}$$

hence the unique solution

$$z(x,t) = z_0(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \left[z_1(\xi) + c\frac{dz_0}{d\xi}(\xi) \right] d\xi$$

For instance, the initial conditions $z(x,0) = z_0(x)$ and $\partial_t z(x,0) \equiv 0$ gives

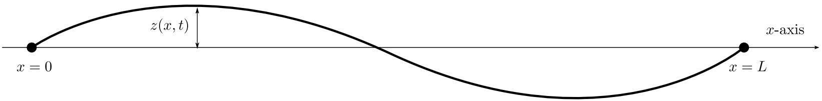
$$z(x,t) = \frac{1}{2} z_0(x-ct) + \frac{1}{2} z_0(x+ct)$$

Boundary conditions, the wave equation example (3/5)

1D wave equation - bounded spatial domain $\mathbf{x} \in [\mathbf{0}, \mathbf{L}]$

state equation: $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$

initial conditions: $\begin{cases} z(x,0) = z_0(x) & \forall \mathbf{x} \in [\mathbf{0}, \mathbf{L}] \\ \frac{\partial z}{\partial t}(x,0) = z_1(x) & \forall \mathbf{x} \in [\mathbf{0}, \mathbf{L}] \end{cases}$



Vibrating string attached at both ends

boundary conditions (Dirichlet): $\begin{cases} z(0,t) = 0 & \forall \mathbf{t} \geq \mathbf{0} \\ z(L,t) = 0 & \forall \mathbf{t} \geq \mathbf{0} \end{cases}$

Boundary conditions, the wave equation example (4/5)

The **initial conditions** define the waveform solutions

$$\begin{cases} \phi(x) = z_0(x) - \psi(x) \\ \psi(x) = \int_0^x \frac{1}{2c} \left[z_1(\xi) + c\frac{dz_0}{d\xi}(\xi) \right] d\xi + A \end{cases}$$

only for $x \in [0, L]$ but the **boundary conditions** read

$$\begin{array}{c} z(0,t) = 0 = \phi(-ct) + \psi(+ct) \\ z(L,t) = 0 = \phi(L-ct) + \psi(L+ct) \end{array}$$

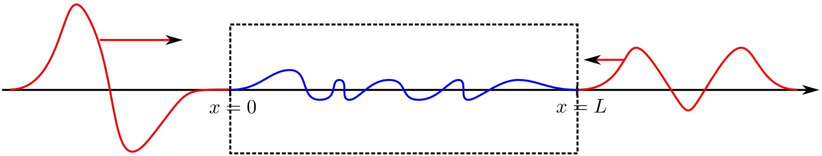
and alllow to **extend the waveforms** ϕ and ψ and the computation of

$$z(x,t) = \phi(x-ct) + \psi(x+ct)$$

for any time $t \geq 0$ and position $x \in [0, L]$ including those with

$$\mathbf{x - ct < 0} \quad \text{or} \quad \mathbf{x + ct > L}$$

Boundary conditions, the wave equation example (5/5)



- **unbounded spatial domain $x \in \mathbb{R}$:** initial conditions given in $[0, \infty)$ allows to predict the behaviour in $[0, L]$ for all time $t > 0$
- **bounded spatial domain $x \in [0, L]$:** boundary conditions are required to predict the behaviour outside $0 \leq x - ct < x + ct \leq L$

some heuristic rules for boundary conditions (1D problems)

- as much as the **higher order of derivation** N_x w.r.t. x
- involving **derivatives up to order $N_x - 1$**
- must **propagate through** the considered spatial domain

Boundary conditions, the heat equation example (1/2)

The **heat equation**

$$\frac{\partial \theta}{\partial t}(t,x) = k\frac{\partial^2 \theta}{\partial x^2}(t,x)$$

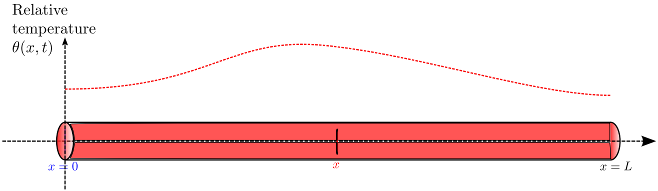
with

- **initial condition:**

$$\theta(x,0) = \theta_0(x)\ \forall x \in [0,L]$$

- **boundary conditions (Neuman):**

$$\begin{aligned}\frac{\partial \theta}{\partial x}(0,t) &= 0\\ \frac{\partial \theta}{\partial x}(L,t) &= 0\end{aligned}$$



Boundary conditions, the heat equation example (2/2)

Assuming symmetry the heat equation in polar coordinates reads:

$$\frac{\partial \theta}{\partial t}(r,t) = k\Delta\theta(r,t) = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}(r,t)\right)$$

Initial condition:

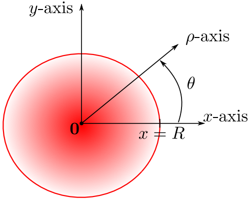
$$\theta(0,x) = 0 \;\; \forall r \in [0,R]$$

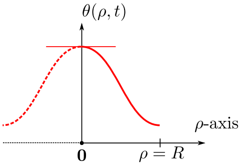
Boundary condition (Neuman):

$$\frac{\partial \theta}{\partial r}(R,t) = 0 \;\; \forall t \geq 0$$

Hidden boundary condition: assuming **smoothness** and **symmetry**

$$\frac{\partial \theta}{\partial r}(0,t) = 0 \; \forall t \geq 0$$





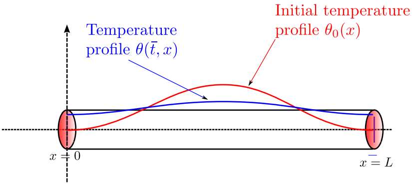
State space definition (1/2)

The **heat equation**

$$\frac{\partial \theta}{\partial t}(t,x) = k\frac{\partial^2 \theta}{\partial x^2}(t,x)$$

with **Neuman** boundary conditions:

$$\begin{aligned} \frac{\partial \theta}{\partial x}(0,t) &= 0 \\ \frac{\partial \theta}{\partial x}(L,t) &= 0 \end{aligned}$$

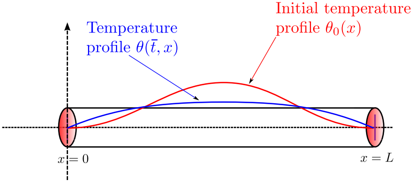


The **heat equation**

$$\frac{\partial \theta}{\partial t}(t,x) = k\frac{\partial^2 \theta}{\partial x^2}(t,x)$$

with **Dirichlet** boundary conditions:

$$\begin{aligned} \theta(0,t) &= 0 \\ \theta(L,t) &= 0 \end{aligned}$$



State space definition (2/2)

- The **initial state** (temperature profile) at time $t_0 = 0$

$$\theta_0 : [0, L] \to \mathbb{R} : x \mapsto \theta_0(x)$$

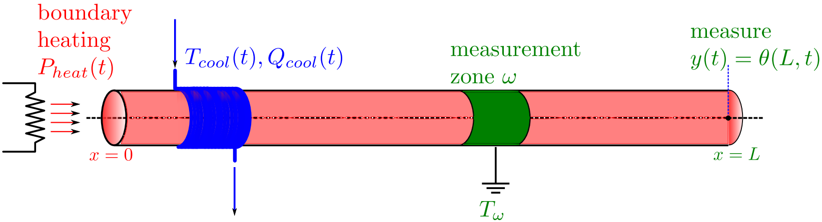
allows to compute the **trajectory** $\theta(t)$ (solution) for $t > 0$

- The state $\theta(t)$ is the (unique) solution of both:
  - the **state equation** $\frac{\partial \theta}{\partial t}(t, x) = k \frac{\partial^2 \theta}{\partial x^2}(t, x)$
  - the **boundary conditions** $\begin{cases} \frac{\partial \theta}{\partial x}(0, t) = 0 \\ \frac{\partial \theta}{\partial x}(L, t) = 0 \end{cases}$
- The **state space**, e.g.

$$Z := H_2^0([0, L]) = \left\{ h : [0, L] \to \mathbb{R} \;\middle|\; \begin{array}{l} h, \frac{dh}{dx}, \frac{d^2 h}{dx^2} \in L_2([0, L]) \\ \frac{dh}{dx}(0) = 0 \\ \frac{dh}{dx}(L) = 0 \end{array} \right\}$$

is equiped with some relevant structure (e.g. inner product and/or norm)

Control systems (1/3)



$$
\begin{aligned}
\frac{\partial \theta}{\partial t}(t,x) &= k \frac{\partial^2 \theta}{\partial x^2}(t,x) + \alpha(\theta_\omega(T_{cool}(t), Q_{cool}(t), x) - \theta(t,x)) \\
T_\omega(t) &= \frac{1}{|\omega|} \int_\omega \theta(t,x)\, dx
\end{aligned}
$$

with **finite rank bounded** linear output $T_\omega(t)$ and **distributed** control

$$\theta_\omega(t,x) = \theta_\omega(T_{cool}(t), Q_{cool}(t), x) \text{ with } (T_{cool}(t), Q_{cool}(t)) \in \mathbb{R}^2$$

and **boundary** control/measurement:

$$-D_t \frac{\partial \theta}{\partial x}(0,t) = P_{heat}(t) =: u_\partial(t) \text{ and } y_\partial(t) = \theta(t, L)$$

Control systems (2/3)

$$\begin{aligned}
\frac{\partial\theta}{\partial t}(t,x) &= k\frac{\partial^2\theta}{\partial x^2}(t,x) + \alpha(T_{wall}(t,x) - \theta(t,x)) \\
&= k\frac{\partial^2\theta}{\partial x^2}(t,x) - \alpha\theta(t,x) + Bu(t)
\end{aligned}$$

with **finite rank bounded** linear **distributed** control:

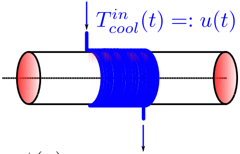
$$B : U := \mathbb{R} \to Z : u(t) \mapsto (Bu)(t,x) := \psi(x)u(t)$$

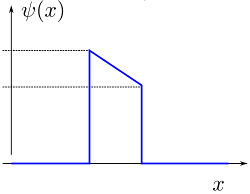
and

$$\|Bu(t)\|_Z = |u(t)|\,\|\psi(\cdot)\|_Z \Rightarrow \|B\|_{\mathcal{L}(U,Z)} = \|\psi\|_Z < \infty$$

Remark: spatial distribution of the control action often has an "intuitive" physical meaning

$T_{wall}(x,t) := u(t)\psi(x)$





Control systems (3/3)

Assume $Z \subset L_2([0, L])$, then the output operator

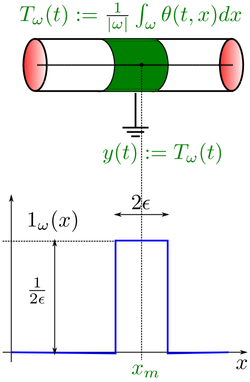
$$C : Z \to Y := \mathbb{R} : z \mapsto \frac{1}{|\omega|}\int_\omega z(x)dx$$

satisfies

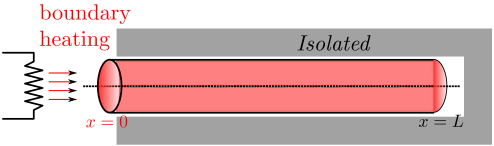
$$\begin{aligned}
|Cz| &= \frac{1}{|\omega|}\left|\int_\omega z(x)dx\right| \\
&\leq \frac{1}{2\epsilon}\int_\omega |z(x)|dx \\
&= \int_{[0,L]} 1_\omega(x)|z(x)|dx \\
&\leq \|1_\omega\|_{L_2([0,L])}\cdot\|z\|_{L_2([0,L])}
\end{aligned}$$

Hence

$$\|C\|_{\mathcal{L}(L_2,\mathbb{R})} \leq \frac{1}{\sqrt{2\epsilon}} < \infty$$



Homogenization (1/3)



We consider the heat equation

$$\frac{\partial \theta}{\partial t}(t,x) \quad = \quad \mathcal{A}\theta(t,x) = k\frac{\partial^2 \theta}{\partial x^2}(t,x)$$

with **boundary control**

$$\mathcal{B}\theta := \left(\begin{array}{c} \frac{\partial \theta}{\partial x}(0,t) \\ \frac{\partial \theta}{\partial x}(L,t) \end{array} \right) = u_\partial(t) = \left(\begin{array}{c} u_\partial^0(t) \\ u_\partial^L(t) \end{array} \right)$$

Homogenization (2/3)

The change of state variable

$$z(t,x) := \theta(t,x) - B(x)u_\partial(t)$$

with $B : [0,L] \to \mathbb{R}^2 : x \mapsto [B_1(x); B_2(x)]$ such that

$$\mathcal{B}B(x)u_\partial = u_\partial$$

transforms the original boundary control problem into

$$\begin{aligned}
\frac{\partial z}{\partial t}(t,x) &= \mathcal{A}z(t,x) + \underbrace{\mathcal{A}B(x)u_\partial(t) - B(x)\dot{u}_\partial(t)}_{\tilde{u}(t,x)} \\
\frac{\partial z}{\partial x}(0,t) &= 0 \\
\frac{\partial z}{\partial x}(L,t) &= 0
\end{aligned}$$

with finite rank **distributed control** $\tilde{u}(t,x)$ and **homogeneous boundary conditions**

Homogenization (3/3)

In the previous heat equation example

$$B(x) = [B_1(x); B_2(x)] := \left[x\left(1 - \frac{x}{2L}\right) \ ; \ \frac{x^2}{2L} \right]$$

gives

$$\mathcal{B}B(x)u_\partial = \begin{pmatrix} \left.\frac{\partial B(x)}{\partial x}\right|_{x=0} \\ \left.\frac{\partial B(x)}{\partial x}\right|_{x=L} \end{pmatrix} u_\partial = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_\partial = u_\partial$$

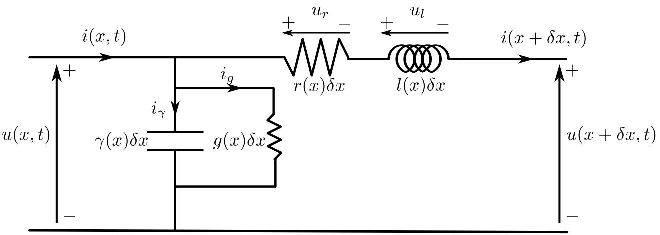
and $z := \theta - Bu_\partial$ transforms the original boundary control problem into

$$\begin{aligned} \frac{\partial z}{\partial t}(t,x) &= k\frac{\partial^2 z}{\partial x^2}(t,x) + \left[\frac{-1}{L}; \frac{+1}{L}\right] \begin{bmatrix} u_\partial^0 \\ u_\partial^L \end{bmatrix} - \left[x\left(1 - \frac{x}{2L}\right) \ ; \ \frac{x^2}{2L} \right] \begin{bmatrix} \dot{u}_\partial^0 \\ \dot{u}_\partial^L \end{bmatrix} \\ \frac{\partial z}{\partial x}(0,t) &= 0 \\ \frac{\partial z}{\partial x}(L,t) &= 0 \end{aligned}$$

and $u_\partial^L(t) \equiv 0$ (homogeneous boundary condition at the isolated end $x = L$)

Building DPS models from local lumped balance equations in infinitesimal volumes

The telegrapher's equation example (1/4)



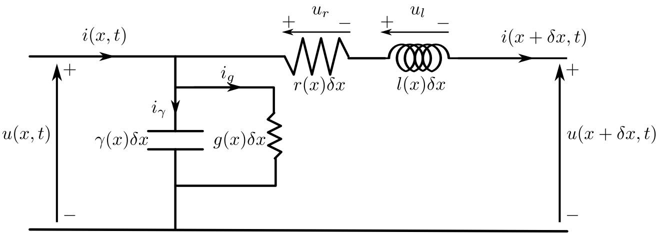
Constitutive equations:

$$\begin{aligned} u_r &= r(x)\,\delta x\, i\,(x+\delta x,t) \\ u_l &= \frac{\partial}{\partial t}\left(l(x)\,\delta x\, i\,(x+\delta x,t)\right) \\ i_g &= g(x)\,\delta x\, u(x,t) \\ u_l &= \frac{\partial}{\partial t}\left(\gamma(x)\,\delta x\, u(x,t)\right) \end{aligned}$$

Balance equations (Kirchoff's laws):

$$\begin{aligned} u(x+\delta x,t) &= u(x,t) - u_r - u_l \\ i(x+\delta x,t) &= i(x,t) - i_\gamma - i_g \end{aligned}$$

The telegrapher's equation example (2/4)



Balance equations:

$$\begin{aligned} u(x,t) + \frac{\partial u}{\partial x}(x,t)\delta x + o(\delta x) &= u(x,t) - r\,\delta x\, i(x+\delta x,t) - l\,\delta x \frac{\partial i}{\partial t}(x+\delta x,t) \\ i(x,t) + \frac{\partial i}{\partial x}(x,t)\delta x + o(\delta x) &= i(x,t) - g\,\delta x\, u(x,t) - \gamma\,\delta x \frac{\partial u}{\partial t}(x,t) \end{aligned}$$

Considering $\delta x \to 0$:

$$\begin{aligned} \frac{\partial u}{\partial x}(x,t) &= -r(x)i(x,t) - l(x)\frac{\partial i}{\partial t}(x,t) \\ \frac{\partial i}{\partial x}(x,t) &= -g(x)u(x,t) - \gamma(x)\frac{\partial u}{\partial t}(x,t) \end{aligned}$$

The telegrapher's equation example (3/4)

In the uniform case (i.e. $l(x) = l, \gamma(x) = \gamma, r(x) = r, g(x) = g$), differentiating these balance equations, one gets:

$$\begin{aligned}
l\gamma \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + (r\gamma + gl)\frac{\partial u}{\partial t}(x,t) + gru(x,t) &= 0 \\
l\gamma \frac{\partial^2 i}{\partial t^2}(x,t) - \frac{\partial^2 i}{\partial x^2}(x,t) + (r\gamma + gl)\frac{\partial i}{\partial t}(x,t) + gri(x,t) &= 0
\end{aligned}$$

which are **hyperbolic** second order PDEs since $-l\gamma < 0$.

For $r = g = 0$ we get:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

with

$$c = \frac{1}{\sqrt{l\gamma}}$$

The telegrapher's equation example (4/4)

Interactions with the environment (**boundary conditions**):

- boundary $x = 0$

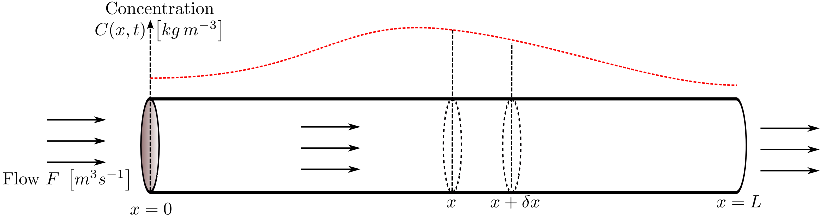
$$u(0,t) = e(t)$$

- boundary $x = L$

$$\begin{cases} \frac{dQ_{out}}{dt}(t) = i(L,t) \\ u(L,t) = R_{out}\, i(L,t) + \frac{Q_{out}(t)}{C_{out}} \\ Q_{out}(0) = Q_0 \end{cases}$$

$\rightarrow$ the transmission line may be connected to any other kind of lumped (or distributed) active or passive circuit

The reaction - advection - dispersion equation (1/5)



Mass balance equation for the $[x, x+\delta x]$ "element":

$$\begin{aligned}\frac{\partial}{\partial t}[S\,\delta x\,C(x,t)] &= [FC(x,t) - FC(x+\delta x,t)] \ \textbf{advection} \\ &+ \left[-D_m S\frac{\partial C}{\partial x}(x,t) + D_m S\frac{\partial C}{\partial x}(x+\delta x,t)\right] \ \textbf{dispersion} \\ &+ S\,\delta x\, r\,(C(x,t)) \ \textbf{reaction}\end{aligned}$$

with S $[m^2]$ the cross section area, D_m $[m^2 s^{-1}]$ the axial dispersion coefficient and $r(C)$ $[kg\, s^{-1} m^{-3}]$ the reaction rate.

The reaction - advection - dispersion equation

(2/5) Using again Taylor's expansions:

$$\begin{aligned} FC(x,t) - FC(x+\delta x,t) &= -F\frac{\partial C}{\partial x}(x,t)\delta x + o(\delta x) \\ -D_m S\frac{\partial C}{\partial x}(x,t) + D_m S\frac{\partial C}{\partial x}(x+\delta x,t) &= +D_m S\frac{\partial^2 C}{\partial x^2}(x,t)\delta x + o(\delta x) \end{aligned}$$

one gets when $\delta x \to 0$:

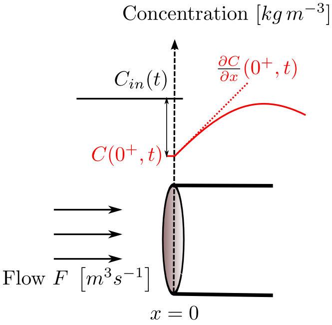
$$\frac{\partial}{\partial t}[S\,C(x,t)] = -F\frac{\partial C}{\partial x}(x,t) + D_m S\frac{\partial^2 C}{\partial x^2}(x,t) + S\,r(C(x,t))$$

Reaction-advection-dispersion parabolic equation

$$\frac{\partial C}{\partial t}(x,t) = -v\frac{\partial C}{\partial x}(x,t) + D_m\frac{\partial^2 C}{\partial x^2}(x,t) + r(C(x,t))$$

with $v := \frac{F}{S}$ $[ms^{-1}]$ the *superficial velocity*

The reaction - advection - dispersion equation (3/5)



Mass conservation at the inlet:

$$FC(0^+, t) - D_m S \frac{\partial C}{\partial x}(0^+, t) = FC_{in}(t)$$

Thermodynamical equilibrium at the outlet:

$$\frac{\partial C}{\partial x}(L^-, t) = 0$$

Danckwertz boundary conditions

$$D_m \frac{\partial C}{\partial x}(0^+, t) = v\left(C(0^+, t) - C_{in}(t)\right)$$

$$\frac{\partial C}{\partial x}(L^-, t) = 0$$

The reaction - advection - dispersion equation (4/5)

Linear reaction - advection - dispersion equation

Assume e.g. a radioactive tracer with a decay $r(C) := -\lambda C$, then:

$$\frac{\partial C}{\partial t}(x,t) \quad = \quad -v\frac{\partial C}{\partial x}(x,t) + D_m \frac{\partial^2 C}{\partial x^2}(x,t) - \lambda C(x,t)$$

With the change of state variable $Z(t,x) := C(t,x)e^{\lambda t}$:

$$\frac{\partial Z}{\partial t}(x,t) = e^{\lambda t}\left(\frac{\partial C}{\partial t}(x,t) + \lambda\left(C(x,t)\right)\right) = -v\frac{\partial Z}{\partial x}(x,t) + D_m \frac{\partial^2 Z}{\partial x^2}(x,t)$$

With $x := \xi + v\tau$ and $t = \tau$, assume $\tilde{Z}(\xi,\tau) := Z(\xi + v\tau, \tau) = Z(x,t)$:

$$\frac{\partial \tilde{Z}}{\partial \tau}(\xi,\tau) = \frac{\partial Z}{\partial t}(\xi+v\tau,\tau)+v\frac{\partial Z}{\partial x}(\xi+v\tau,\tau) = D_m \frac{\partial^2 Z}{\partial x^2}(\xi+v\tau,\tau) = D_m \frac{\partial^2 \tilde{Z}}{\partial \xi^2}(\xi,\tau)$$

The reaction - advection - dispersion equation (5/5)

Non linear transport-reaction-diffusion

Assume a biological tracer with a with logistic growth rate $r(C) := rC\left(1 - \frac{C}{K}\right)$ then:

$$\frac{\partial C}{\partial t}(x,t) = -v\frac{\partial C}{\partial x}(x,t) + D_m \frac{\partial^2 C}{\partial x^2}(x,t) + rC\left(1 - \frac{C}{K}\right)$$

multiple species reaction $aA + bB \rightleftharpoons cC + dD$

Assume reaction kinetics rate $r(A,B)$, then:

$$\begin{aligned}
\frac{\partial A}{\partial t}(x,t) &= -v\frac{\partial A}{\partial x}(x,t) + D_m \frac{\partial^2 A}{\partial x^2}(x,t) - a\, r(A,B) \\
\frac{\partial B}{\partial t}(x,t) &= -v\frac{\partial B}{\partial x}(x,t) + D_m \frac{\partial^2 B}{\partial x^2}(x,t) - b\, r(A,B)
\end{aligned}$$

The reaction - advection - dispersion equation

Examples of reaction kinetics:

- logistic growth reaction rate (saturation)

$$r(X) = \rho \left(1 - \frac{X}{K}\right) X$$

  where X is the population concentration

- Monod reaction rate (biochemical reaction) for $X + k_1 S \rightarrow k_2 X$

$$r(S, X) = \frac{\mu_{max} S}{K_S + S} X$$

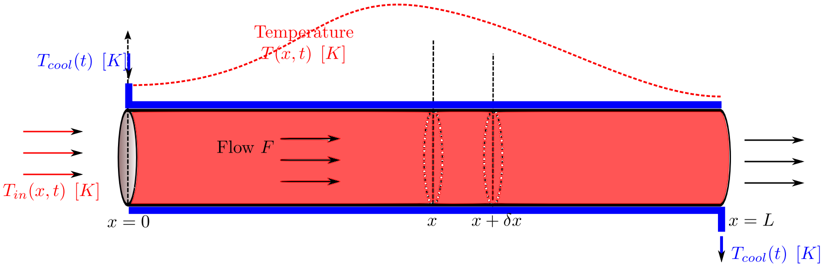
  with S and X the limiting substrate and biomass concentrations

- law of mass action for chemical reactions $aA + bB \rightleftharpoons cC + dD$

$$r(A, B) = k A^a B^b$$

  with for instance $k = k_0 e^{\frac{-E}{RT}}$ (Arrhenius law)

Non isothermal reactor (1/3)



- **external heat** transfer (cooling jacket)

$$U = -D\,\delta x\,R_{th}\,(T(x,t) - T_{cool}(t))$$

with R_{th} and D the wall heat transfer coefficient and diameter ($S = \frac{\pi D^2}{4}$)

- **reaction heat**

$$S\,\delta x\,r\,(C(x,t), T(x,t))\,\Delta H_R$$

with ΔH_R $[J\,kg^{-1}]$ the reaction enthalpy ($\Delta H_R < 0$ for exothermic reaction) and $r(C,T)$ $[kg\,s^{-1}\,m^{-3}]$ the reaction rate.

Non isothermal reactor (2/3)

Heat balance equation for the $[x, x+\delta x]$ "element":

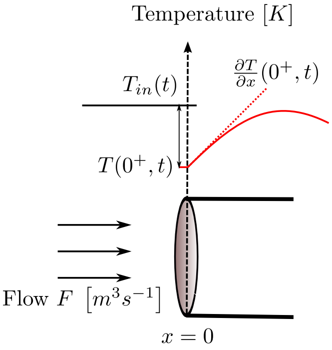
$$
\begin{aligned}
\frac{\partial}{\partial t}[\rho C_p S\,\delta x\,T(x,t)] \quad = \quad & -F\rho C_p \frac{\partial T}{\partial x}(x,t)\,\delta x \ \textbf{convection} \\
& + D_{th} S \frac{\partial^2 T}{\partial x^2}(x,t)\,\delta x \ \textbf{dispersion} \\
& + S\,\delta x\,\Delta H_R r(C,T) \ \textbf{reaction heat} \\
& + D\,\delta x\,R_{th}(T_{cool}(t) - T(x,t)) \ \textbf{exchange with wall}
\end{aligned}
$$

with ρ $[kg\,m^{-3}]$ the specific mass and C_p $[J\,K^{-1}\,kg^{-1}]$ the specific heat.

Mass and heat balance equations

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= -v\frac{\partial C}{\partial x} + D_m \frac{\partial^2 C}{\partial x^2} + r(C,T) \\
\frac{\partial T}{\partial t} &= -v\frac{\partial T}{\partial x} + \frac{D_{th}}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{\Delta H_R}{\rho C_p} r(C,T) + \frac{4R_{th}}{\pi D \rho C_p}(T_{cool}(t) - T)
\end{aligned}
$$

Non isothermal reactor (3/3)



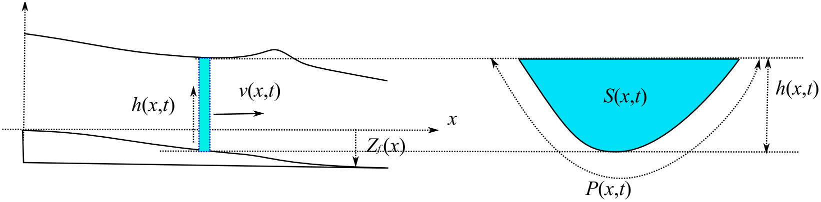
- Mass and **heat (?) conservation** at the inlet
- Thermodynamical equilibrium at the outlet

Danckwertz boundary conditions

$$
\begin{aligned}
D_m \frac{\partial C}{\partial x}(0^+, t) &= v\left(C(0^+, t) - C_{in}(t)\right) \\
D_{th} \frac{\partial T}{\partial x}(0^+, t) &= v\left(T(0^+, t) - T_{in}(t)\right) \\
\frac{\partial C}{\partial x}(L^-, t) &= 0 \\
\frac{\partial T}{\partial x}(L^-, t) &= 0
\end{aligned}
$$

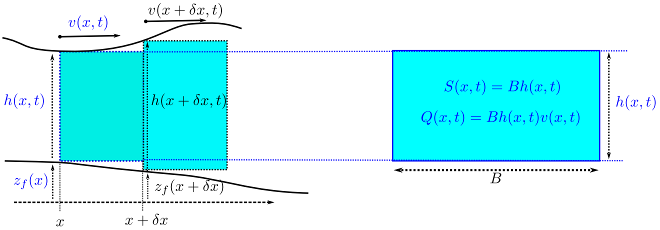
Some examples of free surface shallow water flow models

Free surface fluid flow model (1/5)



- 1D assumption: average velocity field $v(x,t)$ only depends on longitudinal coordinate x
- free surface flow delimited by boundary conditions (gates, weirs, junctions, etc.)
- gravity forces, waveform restoring forces, friction forces, eventually dispersion

Free surface fluid flow model (2/5)

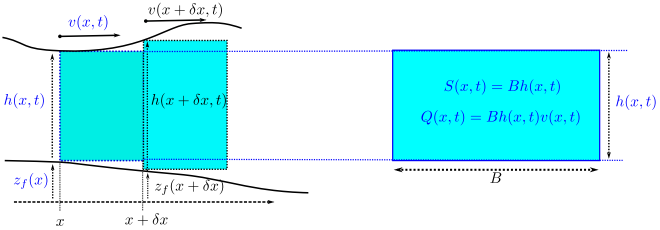


- **Mass balance equation**

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho S(x,t)\delta x\right] &= +\rho S(x,t)v(x,t) - \rho S(x+\delta x,t)v(x+\delta x,t)\\ &= -\frac{\partial}{\partial x}\left(\rho S(x,t)v(x,t)\right) + o\left(\delta x\right)\end{aligned}$$

$$\frac{\partial S}{\partial t}(x,t) = -\frac{\partial Q}{\partial x}(x,t) \;\text{ or }\; \frac{\partial h}{\partial t} = -\frac{\partial (hv)}{\partial x}$$

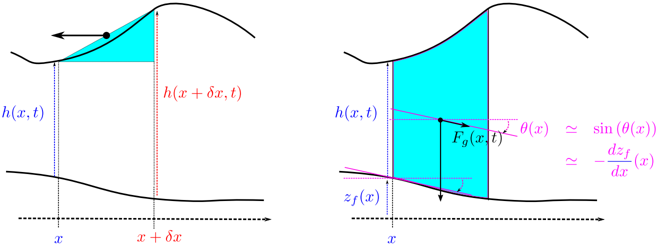
Free surface fluid flow model (3/5)



- **Momentum balance equation**

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho S(x,t)v(x,t)\delta x\right] &= \left[\rho Q(x,t)v(x,t) - \rho Q(x+\delta x,t)v(x+\delta x,t)\right] \\ &+ \text{gravity} + \text{pressure} + \text{external friction} \\ &+ \text{internal dissipation} + \text{wave restoring forces} \\ &+ \text{etc.}\end{aligned}$$

Free surface fluid flow model (4/5)



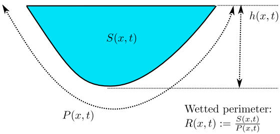
- Force resulting from the **"hydrostatic" pressure** distribution

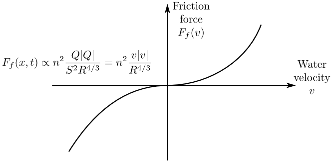
$$P(x,t) = \rho g h(x,t) \;\Rightarrow\; F_p = -S(x,t)\,\delta x\,\frac{\partial P}{\partial x}(x,t) = -\rho g S(x,t)\frac{\partial h}{\partial x}(x,t)\,\delta x$$

- **"Resulting" gravity force** (small slope assumption)

$$F_g \;=\; \sin(\phi)\,\rho S(x,t)\,\delta x\, g \simeq -\frac{dz_f}{dx}(x)\rho S(x,t)\,\delta x\, g$$

Free surface fluid flow model (5/5)





- **"External" friction** forces (Manning Strickler formulae)

$$F_f(x,t) = -\rho S(x,t)\,\delta x\, g\, \underbrace{n^2 \frac{Q(x,t)|Q(x,t)|}{S^2(x,t) R^{4/3}(x,t)}}_{\text{friction slope}}$$

- **"Internal" dissipation** (empirical dispersion/mixing)

$$F_v(x,t) = \rho S(x,t)\,\delta x D_v \frac{\partial^2 v}{\partial x^2}(x,t)$$

Example: the Saint-Venant or Shallow Water Equations (SWE)

- hydrostatic pressure distribution, gravity and Manning-Strickler friction

SWE in (Q, S) variables (waterbed profile $S = S(h, x)$)

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= -\frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{Q^2}{S}\right] - gS\left[\frac{\partial h}{\partial x} + \frac{dz_f}{dx} + n^2 \frac{Q|Q|}{S^2 R^{4/3}}\right]
\end{aligned}
$$

SWE in (h, v) variables (rectangular waterbed $S = Bh$ and $Q = Bhv$)

$$
\begin{aligned}
\frac{\partial h}{\partial t} &= -\frac{\partial (hv)}{\partial x} \\
\frac{\partial v}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{v^2}{2}\right] + g\left[\frac{\partial h}{\partial x} + \frac{dz_f}{dx} + n^2 \frac{v|v|}{\left(\frac{Bh}{B+2h}\right)^{4/3}}\right]
\end{aligned}
$$

Example: the Burger's Equation (1/3)

- Pressure gravity and exterior friction forces are neglected

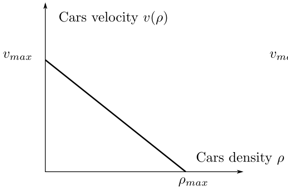
For a rectangular waterbed

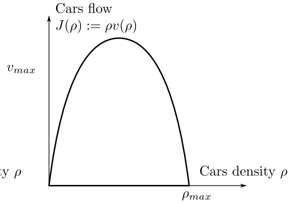
$$\begin{aligned}\frac{\partial h}{\partial t} &= -\frac{\partial (hv)}{\partial x}\\ \frac{\partial v}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{v^2}{2}\right] + D_v \frac{\partial^2 v}{\partial x^2}\end{aligned}$$

Inviscid Burger's equation (dispersion is neglected)

$$\begin{aligned}\frac{\partial h}{\partial t} &= -\frac{\partial (hv)}{\partial x}\\ \frac{\partial v}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{v^2}{2}\right]\end{aligned}$$

Example: the Burger's Equation (2/3)



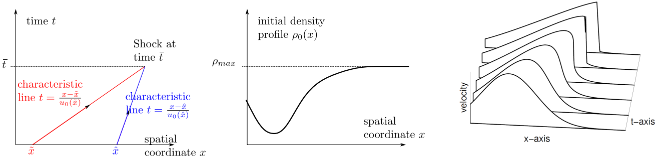


- model for traffic flow of cars on a highway
- density $\rho(x,t)$ and velocity $v(x,t)$ of cars in $x \in \mathbb{R}$ at time $t \geq 0$
- the cars conservation equations reads

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}(x,t) \text{ with } J = \rho v(\rho) = \rho v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

- the solution exhibits **shock formation** (traffic jam) and travelling shocks

Example: the Burger's Equation (3/3)



Assume L and τ are such that $\frac{L}{\tau} = v_{max}$, then the change of variables

$$\tilde{x} := \frac{x}{L} \quad \tilde{t} := \frac{t}{\tau} \quad u := 1 - \frac{2\rho}{\rho_{max}}$$

transforms the previous equation into the Burger equation

$$\frac{\partial u}{\partial \tilde{t}} = -\frac{\partial}{\partial \tilde{x}}\left[\frac{u^2}{2}\right] \quad \text{or} \quad \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{u^2}{2}\right]$$

whose general solution may be obtained by the method of characteristics:

$$u(x + tu_0(x), t) = u_0(x)$$

Others shallow water models

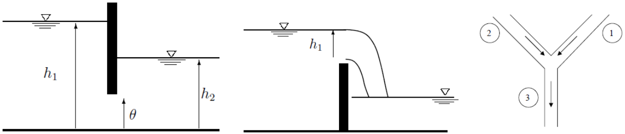
- **Korteweg - De Vries** (Burger's equation with "wave restoring" force)

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}\left[\underbrace{\frac{v^2}{2}}_{\text{steepening}} + \underbrace{\alpha\frac{\partial^2 v}{\partial x^2}}_{\text{spreading}}\right]$$

- **Boussinesq** (SWE with supplementary "dispersive" term)

$$\begin{aligned}
\frac{\partial h}{\partial t} &= -\frac{\partial (hv)}{\partial x} + \frac{h^3}{6}\frac{\partial^3 v}{\partial x^3} \\
\frac{\partial v}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{v^2}{2} + gh\right] + \frac{h^2}{2}\frac{\partial}{\partial t}\frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

Boundary conditions for shallow water equations



Submerged sluice gate — Free weir — Y junction

- Submerged sluice gate

$$Q_1(L,t) = Q_2(0,T) = \alpha B\theta(t)\sqrt{2g\,(h_1(L,t) - h_2(0,T))}$$

- Free weir

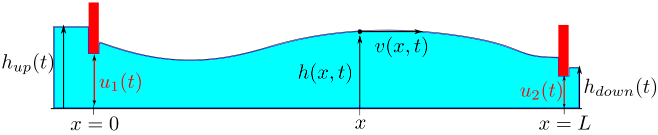
$$Q_1(L,t) = \tilde{\alpha} B h_1(L,t)\sqrt{2gh_1(L,t)}$$

- Y junction

$$Q_1(L,t) + Q_2(L,t) = Q_3(0,t)$$
$$h_1(L,t) = h_2(L,t) = h_3(0,t)$$

Example of control problems for distributed parameters systems

Exact controllability (1/2)

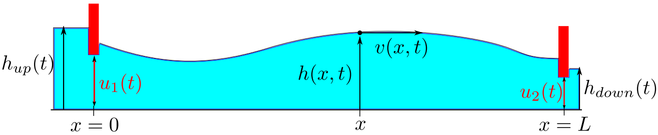


$$
\begin{aligned}
\frac{\partial h}{\partial t}(x,t) &= -\frac{\partial}{\partial x}\left(h(x,t)v(x,t)\right) \\
\frac{\partial v}{\partial t}(x,t) &= -\frac{\partial}{\partial x}\left(gh(x,t) + \frac{v^2(x,t)}{2}\right)
\end{aligned}
$$

Boundary control:

$$
\begin{aligned}
Q(0,t) &:= B(hv)|_{x=0} = \alpha u_1(t)\sqrt{h_{up}(t) - h(0,t)} \\
Q(L,t) &:= B(hv)|_{x=L} = \alpha u_2(t)\sqrt{h(L,t) - h_{down}(t)}
\end{aligned}
$$

Exact controllability (2/2)



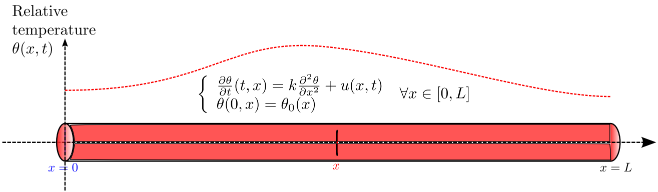
exact controllability

Assume the **initial state profile** is $(h_0, v_0) \in Z$.
The system is **exactly controllable** at $(h_0, v_0) \in Z$ **in time** $T > 0$ if and only if for any admissible **target state profile** $(h_c, v_c) \in Z$,
there exist inputs $u_1(t)$ and $u_2(t)$ admissible on $[0, T]$ such that

$$(h(T), v(T)) \stackrel{Z}{=} (h_c, v_c)$$

where $(h(t), v(t)) \in Z$ denotes the solution for initial conditions (h_0, v_0) and inputs u_1 and u_2

Approximate controllability



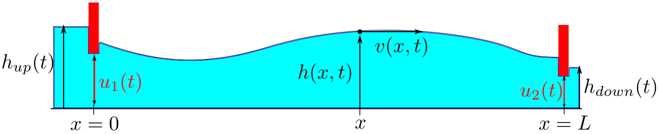
approximate controllability or weak controllability

Assume the initial state profile is $\theta_0 \in Z$.
The system is **approximately controllable** at $\theta_0 \in Z$ in time $T > 0$ iff.
for any admissible target state profile $\theta_c \in Z$ and **for any** $\epsilon > 0$
there exists an input $u(t)$ admissible on $[0, T]$ such that

$$\|\theta(T) - \theta_c\|_Z < \epsilon$$

where $\theta(t) \in Z$ denotes the solution

Regional analysis



regional controllability

Let $\omega \subset \Omega$ be some **region of interest** of the spatial domain. Let the restriction application be defined as:

$$p_\omega : L^2(\Omega) \to L^2(\omega) : z \mapsto p_\omega z := z|_\omega$$

The system is **regionally exactly controllable** on ω at $z_0 \in Z(\Omega)$ in time $T > 0$ iff.
for any admissible target state profile $z_c \in Z(\omega)$ there exists a control $u(t)$ and such that

$$p_\omega z(T) = z_c$$

where $z(t) \in Z(\Omega)$ denotes the solution

Spreadability (1/2)

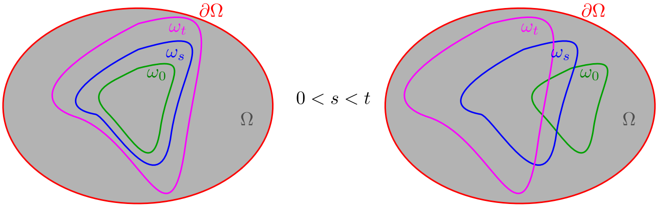
- Examples:
  - desertification / vegetation dynamics / fire spread
  - epidemies / disease spread
  - pollution / convection-diffusion
- Consider a boolean property

$$\begin{array}{rcl} \mathcal{P} : Z & \rightarrow & \{0,1\}^{\Omega} \\ z & \mapsto & (\mathcal{P}z) : \Omega \rightarrow \{0,1\} \\ & & (\mathcal{P}z)(x) = \begin{cases} 1 & \text{when } z(x) \text{ satisfies the property} \\ 0 & \text{else} \end{cases} \end{array}$$

  Examples

  - $(\mathcal{P}z)(x) = 1 \Leftrightarrow z(x) = p(x)$, with p a given profile
  - $(\mathcal{P}z)(x) = 1 \Leftrightarrow z(x) \in K$, with K a set of constraints
- Subregion $\omega_t := \{x \in \Omega \, | (\mathcal{P}z(t))\,(x) = 1\}$ with $z(t)$ the solution
- Assumption: $\omega_0 \neq \emptyset$

Spreadability (2/2)



Spreadability - inclusion   Spreadability - measure

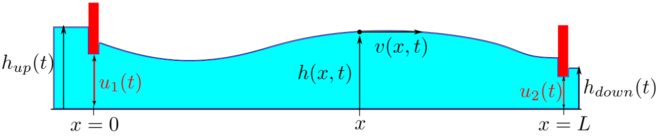
spreadability

The system is $\mathcal{P}$-spreadable from ω_0 during the time interval $I := [0, T]$ if the family $(\omega_t)_{t \in I}$ is non decreasing:

$$\forall s, t \in I : s \leq t \Rightarrow \omega_s \subset \omega_t \text{ or } \mu(\omega_s) \leq \mu(\omega_t)$$

Otherwise it is said $\mathcal{P}$-resorbable

Example stabilizing feedback



Look for static state (operator) feedback

$$\begin{pmatrix} h(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \mapsto u_1(t)$$

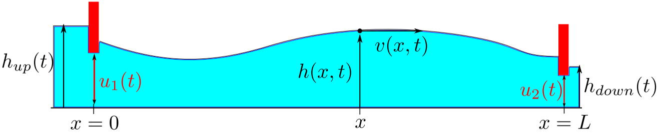
$$\begin{pmatrix} h(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \mapsto u_2(t)$$

such that

$$\left\| \begin{pmatrix} h(\cdot,t) - h_c(\cdot) \\ v(\cdot,t) - v_c(\cdot) \end{pmatrix} \right\|_Z \to 0 \text{ when } t \to \infty$$

where h_c and v_c are target water level and velocity profiles

Example LQ control



Look for control signals $u_1(t)$ and $u_2(t)$, $t \in [0, T]$ which solve

$$\min_{u_1,u_2} \int_0^T \left\| \begin{bmatrix} h(t)-h_c \\ v(t)-v_c \end{bmatrix} \right\|_Q^2 + \left\| \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\|_R^2 dt$$

with ($Q(x) > 0$ a.e. in $[0, L]$)

$$\left\| \begin{bmatrix} h(t)-h_c \\ v(t)-v_c \end{bmatrix} \right\|_Q^2 := \int_0^L \begin{bmatrix} h(t,x)-h_c(x) \\ v(t,x)-v_c(x) \end{bmatrix}^T Q(x) \begin{bmatrix} h(t,x)-h_c(x) \\ v(t,x)-v_c(x) \end{bmatrix} dx$$

and where $(h(t), v(t))^T$ is the solution of the SWE with sluice gates boundary conditions

Classical control problems for DPS

Classical problems such as

- stabilization / state or output feedback control
- optimal control
- estimation / feedback observer design
- frequency-domain description and control
- reduced order control and robustness
- distributed parameters identification

may be addressed with specific techniques

Specific control problems for DPS

Some control problems specifically arise for DPS

- actuators/sensors placement
- localization (perturbations)
- regional control/observation
- spray control
- mobile sensors/actuators
- moving boundary control problems